

# Optimal Liability Allocation under Mortality Parameter Uncertainty: The Conditional Value-at-Risk Approach

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## Abstract

This paper proposes a Conditional Value-at-Risk Minimization (CVM) approach to optimize the allocation of liabilities for insurance companies. By incorporating the natural hedging strategy of Cox and Lin (2007) and the two-factor stochastic mortality model of Cairns et al. (2006b), we calculate an optimal liability allocation to hedge against the systematic mortality risk under parameter uncertainty. To reflect the importance of required profit, we further integrate the premium loadings of systematic risk into the proposed approach. We compare the hedging results to those using the duration match method of Wang et al. (2009) and show that the proposed CVM approach has a narrower quantile of liability distribution after hedging and thus effectively reduces systematic mortality risks for life insurance companies.

**Keywords:** mortality systematic risk, liability management, natural hedging, parameter risk, Conditional VaR.

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# 1 Introduction

Over the past decade, a longevity shock has spread across human society. Benjamin and Soliman (1993) and McDonald et al. (1998) confirm that unprecedented improvements in population longevity have occurred worldwide. The decreasing trend in the mortality rate has created a great risk for insurance company. Existing literature has proposed many solutions to mitigate the threat of longevity risk to life insurance companies. We can classify these solutions into three categories. The *capital market solutions*, include mortality securitization (see, for example, Dowd 2003; Lin and Cox 2005; Blake et al. 2006a, 2006b; Cox et al. 2006), survivor bonds (e.g., Blake and Burrows 2001; Denuit et al. 2007), and survivor swaps (e.g., Dowd et al. 2006). These studies suggest that insurance companies could transfer their exposures to the capital markets. Cowley and Cummins (2005) provide an excellent overview of the securitizations of life insurance assets and liabilities. The second solutions, the *industry self-insurance solutions*, include the natural hedging strategy of Cox and Lin (2007), the duration matching strategy of Wang et al. (2009), and the reinsurance swap of Lin and Cox (2005). The advantages of these solutions are that the hedging does not require a liquid market and can be arranged at a lower transaction cost. Insurance companies can hedge longevity risk by themselves or with their counterparties. The third solution, the *mortality projection improvement*, provides a more accurate estimation of mortality processes. As Blake et al. (2006b) propose, these studies fall into two areas: continuous-time frameworks (e.g., Milevsky and Promislow 2001; Dahl 2004; Biffis 2005; Schrager 2006) and discrete-time frameworks (e.g., Brouhns et al. 2002; Renshaw and Haberman 2003; Cairns et al. 2006b). The parameter uncertainty and model specification in relation to the mortality process have also attracted more attentions in recent years.

Among the industry self-insurance solutions, the natural hedging strategy suggests that life insurance can serve as a hedging vehicle against longevity risk for annuity products. Wang et al. (2009) employ duration as a measure of the liability sensitivity to mortality change and propose a mortality duration matching (MDM) approach to calculate the optimal product mix. However, their work contains several restricted assumptions. First, they assume that future mortality changes involve parallel shifts in the mean and do not measure the higher-order moment risk of mortality distribution. Second, the MDM approach applies only to two liabilities. Third, the MDM approach is a pure risk reduction method because the profit loading is not considered during the hedging procedure. Fourth, Melnikov and Romaniuk (2006) and Koissi et al. (2006) suggest that parameter risk is crucial when dealing with longevity risk. The parameter uncertainty does not play a role in the MDM approach since they consider the mortality shift only in terms of its mean.

To overcome the above problems, we employ the two-factor stochastic mortality model of Cairns et al. (2006b) and construct the Conditional Value-at-Risk Minimization (CVM) approach to control the possible loss. Managing liabilities risk with parameter uncertainty is one feature of the CVM approach. The other feature is that we add the profit loading constraint in the optimization. The premium-pricing principle suggested by Milevsky et al. (2006) is employed to estimate the required profit loadings – to compensate the stockholders bearing

systematic mortality risk with the same Sharpe ratio as other asset classes in the economy.<sup>1</sup> Furthermore, the CVM approach could be easily implemented using linear programming (Rockafellar and Uryasev, 2000) and insurance companies can adopt it as their own internal risk management tool.

In the results of our simulation, we find that the proposed CVM approach has a narrower quantile of liability distribution after hedging and thus effectively reduces systematic mortality risks for life insurance companies. On the other hand, the MDM approach has a limited hedging effect in distribution risk. In addition, the CVM approach considers not only risk reduction but also the required profit constraint. We find that the required loading substantially changes the optimal allocations of liabilities and could not be ignored.

The remainder of this article is organized as follows: The models are outlined in Section 2, including the mortality model of Cairns et al. (2006b), the duration matching strategy of Wang et al. (2009), the loading estimation of Milevsky et al. (2006), and the CVM approach. In Section 3, we introduce the mortality data, project future mortality and design the liabilities. In Section 4, we present the numerical examples in two scenarios: two-liability scenario without a required loading constraint and a multiple liabilities scenario with a required loading constraint. We compare the hedging results of the MDM and CVM approaches in this section. We close with some conclusions and implications in Section 5.

## 2 The Models

We first briefly review the two-factor stochastic mortality model of Cairns et al. (2006b), the mortality duration matching of Wang et al. (2009), and the loading estimation methods of Milevsky et al. (2006). We then introduce the CVM approach.

### 2.1 The Two-Factor Stochastic Mortality Model

Several stochastic models proposed in the literature attempt to capture the processes of the mortality rate. We choose the two-factor mortality model (the CBD model) as the underlying mortality process for two reasons. First, the CBD model characterizes not only a cohort effect but also a quadratic age effect. The two factors  $A_1(t)$  and  $A_2(t)$  in the CBD model represent all age-general improvements in mortality over time and different improvements for different age groups. These two factors reflect the “trend effect” and “age effect”. Thus, the analysis will be economically or *biologically* meaningful when we consider the parameter changes of these factors over time. Second, the CBD model is a discrete time model and can be more conveniently implemented in practice. We offer a brief description of the two-factor model; for a more detailed discussion, see Cairns et al. (2006b).

Let  $q_{t,x}$  be the realized mortality rate for age  $x$  insured from time  $t$  to  $t + 1$ . Assume the

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<sup>1</sup>The non-systematic risk of liabilities in this paper is not considered. We assume that the non-systematic mortality risks are all diversified across policyholders by the law of large numbers. The shareholders bearing non-systematic mortality risk are not rewarded. We also assume that insurance companies will not suffer from an insolvency problem.

mortality curve has a logistic functional form:

$$q_{t,x} = \frac{e^{A_1(t+1)+A_2(t+1)\cdot(x+t)}}{1 + e^{A_1(t+1)+A_2(t+1)\cdot(x+t)}}. \quad (1)$$

The two stochastic trends  $A_1(t+1)$  and  $A_2(t+1)$  follow a random walk process with drift parameter  $\mu$  and diffusion parameter  $C$ :

$$A(t+1) = A(t) + \mu + CZ(t+1), \quad (2)$$

where  $A(t+1) = [A_1(t+1), A_2(t+1)]^T$  and  $\mu = [\mu_1, \mu_2]^T$  are a  $2 \times 1$  constant parameter vector.  $C$  is a  $2 \times 2$  constant upper-triangular Cholesky square root matrix of the covariance matrix  $V = CC^T$  and  $Z(t)$  is a two-dimensional standard normal random variable. To include the uncertainty of  $\mu$  and  $C$ , Cairns et al. (2006b) invoke a normal-inverse-Wishart distribution from a non-informative prior distribution:

$$V^{-1} | D \sim \text{Wishart}(n-1, n^{-1}\hat{V}^{-1}) \quad (3)$$

$$\mu^{-1} | V, D \sim \text{MVN}(\hat{\mu}, n^{-1}V),$$

where  $D(t) = A(t) - A(t-1)$ ,

$$\hat{\mu} = \frac{1}{n} \sum_{t=1}^n D(t),$$

and  $\hat{V} = \frac{1}{n} \sum_{t=1}^n (D(t) - \hat{\mu})(D(t) - \hat{\mu})^T$ .

Thus, we can generate  $A(t)$  from equation (2) with the parameters  $\mu$  and  $C$  from equation (3). Then we get  $q_{t,x}$  as equation (1) suggests.

## 2.2 The Mortality Duration Matching (MDM) Method

Wang et al. (2009) propose the MDM approach to calculate an optimal life insurance/annuity weight to immunize the value change from mortality risk. In their work, the following product mix of life insurance is proposed:

$$w^D = \frac{D^a}{D^a + D^l}, \quad (4)$$

where  $D^a$  denotes the effective duration of the annuity and  $D^l$  denotes the effective duration of the life insurance. Formally, the effective duration can be calculated as:

$$D^a = -\frac{V^{a+} - V^{a-}}{2V^a\Delta q} \quad \text{and} \quad D^l = \frac{V^{l+} - V^{l-}}{2V^l\Delta q}.$$

The  $\Delta q$  refers to the change in the mortality rate,  $V^{a+}$  and  $V^{l+}$  represent the liability values at higher mortality rate ( $q + \Delta q$ ) and  $V^{a-}$  and  $V^{l-}$  represent the values at the lower mortality

rate  $(q - \Delta q)$ . If the change is small, this strategy leads to the liability immunization as follows:

$$\Delta V = w^D D^l - (1 - w^D) D^a = 0. \quad (5)$$

Wang et al. (2009) also propose the mortality convexity adjustment for a large change as

$$C^a = \frac{V^{a-} + V^{a+} - 2V^a}{V^a(\Delta q)^2} \quad \text{and} \quad C^l = \frac{V^{l-} + V^{l+} - 2V^l}{V^l(\Delta q)^2}.$$

Then the product mix weight with convexity on life insurance is

$$w^C = \frac{D^a - \frac{\Delta q}{2} C^a}{D^a + D^l + \frac{\Delta q}{2} (C^l - C^a)}. \quad (6)$$

Here the change is set as  $\Delta q = \bar{q}(1 + s) - \bar{q}$ , where  $\bar{q}$  is the mean of the mortality process and  $s$  is a shift proportion such as 1%. Thus, the change here involves a parallel shift in the mean.

### 2.3 Estimate Profit Loading: The Sharpe Ratio Method

Milevsky et al. (2006) show that when the mortality rate is stochastic, the standard deviation per policy does not vanish based on the law of large numbers. There instead exists some systematic or market risk in a diversified liability (under a large number of policies). The shareholders of an insurance company request a risk premium for bearing such systematic risk. Milevsky et al. propose that the risk loading  $\pi$ , which is used to compensate shareholders with the Sharpe ratio, equals other asset classes in the economy. The Sharpe ratio for the liability premium is defined as:

$$SR = \frac{E(V)(1 + \pi) - E(V)}{\sigma(V)}, \quad (7)$$

where  $E(V)$  is the expected or actuarial fair price of the liability under the law of large numbers, and  $\sigma(V)$  is the standard deviation of liability values. When the capital market is in equilibrium, we can set equation (7) equal to the Sharpe ratio of some broadly diversified portfolio, such as the S&P500 index, and solve  $\pi$  implicitly. For more details, please see Section 4.2.

### 2.4 The Conditional Value-at-Risk Minimization (CVM) Approach

Let the random variable  $v^i$  be the value of the  $i$ -th liability and its present value or actuarially fair price be  $E(v^i)$ . Owing to the stochasticity of  $q$ ,  $v^i$  will generate a deviation from  $E(v^i)$ . The loss proportions for each liability are denoted as

$$r^i = \frac{v^i - E(v^i)}{E(v^i)}, \quad (8)$$

The total loss proportion is

$$r_p = \sum_i w^i r^i, \quad (9)$$

where  $w^i$  is the weight of the  $i$ -th liability in relation the total liability. The  $i$ -th liability could refer to a life insurance or an annuity. We engage in natural hedging to minimize the risk of  $r_p$  by choosing different  $w^i$ . The Conditional VaR (CVaR) is proposed to measure the liability risk. CVaR is chosen as a risk measure instead of VaR, because CVaR is a coherent measure whereas VaR is not, as shown by Artzner et al. (1997, 1999) and Deprez and Gerber (1985). The CVM approach is expressed as:

$$\mathbf{Min}_{w^i} \quad E[r_p | r_p \geq r_p(\alpha)] \quad (10)$$

$$s.t. \quad \sum_i w^i \cdot \pi^i \geq \bar{\pi}, \quad (11)$$

$$\sum_i w = 1, \text{ and } 0 \leq w^i \leq 1. \quad (12)$$

where  $E[r_p | r_p \geq r_p(\alpha)]$  is the conditional expected loss that exceeds the threshold,  $r_p(\alpha)$ , under the specified probability  $\alpha$ . In equation (11),  $\pi^i$  denotes the profit loading on the  $i$ -th liability charged by the insurance company and is estimated using the Sharpe ratio as mentioned in Section 2.3. The weighted profit  $\sum w^i \cdot \pi^i$  is required to be greater than or equal to  $\bar{\pi}$ . Here we let the target profit,  $\bar{\pi}$ , be exogenously given. We ensure that the sum of the weights is equal to one and prohibit short selling by equation (12). Although CVaR is usually defined in terms of monetary value, here we present it as a percentage loss to avoid confusion over magnitude. Rockafellar and Uryasev (2000) confirm this to be the case when there is a one-to-one correspondence between the percentage return and monetary value.

In the CVM approach,  $r_p$  is generated by means of the following steps. First, we apply the CBD model to simulate the mortality processes and corresponding distributions of  $v^i$ . We compute  $E(v^i)$  and substitute it into equation (8) to obtain the distribution of  $r^i$ . We calculate  $r_p$  by equation (9). It is crucial when we aggregate  $r_p$  to keep all the dependence under the same mortality scenario. The CBD model we employ is able to take the parameter uncertainty into consideration. With this approach, we deal with the longevity risk and parameter uncertainty simultaneously. To demonstrate the results of the optimization, we provide three examples in Section 4.

### 3 Mortality Estimations and Liability Designs

This section describes the mortality data set and liabilities. We employ the data from Cairns et al. (2007) and the JPMorgan LifeMetrics model (2006); a sample of U.S. men aged 60-84 from 1968 to 1979 and U.S. men aged 60-89 from 1980 to 2003.<sup>2</sup> The estimated drift and diffusion parameters are:

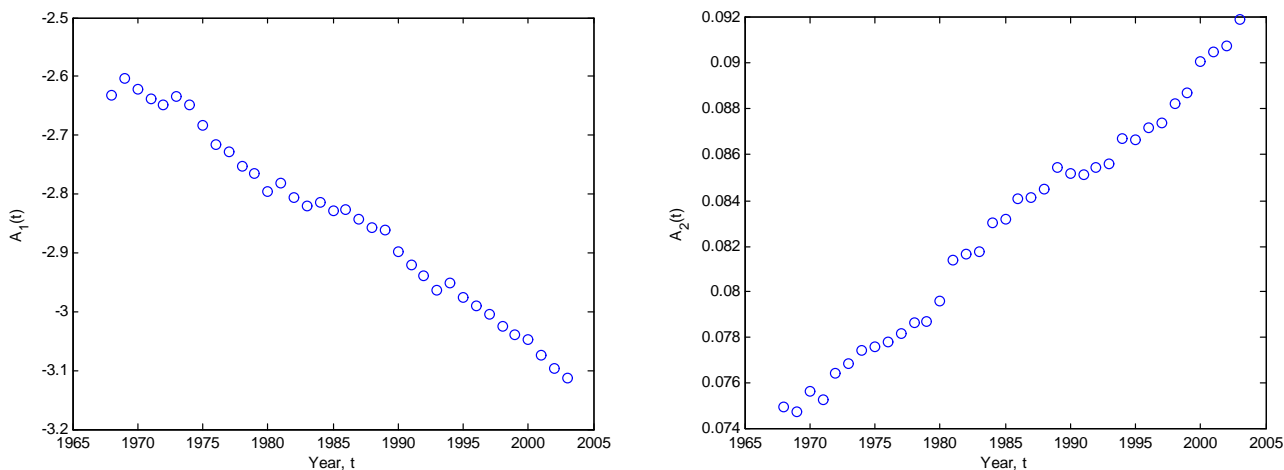
$$\hat{\mu} = \begin{bmatrix} -0.016289 \\ 0.0004769 \end{bmatrix} \text{ and } \hat{V} = \hat{C}\hat{C}^T = \begin{bmatrix} 0.00011695 & 0.00000334 \\ 0.00000334 & 0.00000031 \end{bmatrix}.$$

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<sup>2</sup>The data set and CBD estimation model are available on the JPMorgan LifeMetrics Web site: <http://www.jpmorgan.com/pages/jpmorgan/investbk/solutions/lifemetrics>.

The ‘hat’ indicates the estimation value. By substituting the coefficients  $\hat{\mu}$  and  $\hat{V}$  into equation (2), we obtain  $A(t) = [A_1(t), A_2(t)]^T$ . The paths of  $A_1(t)$  and  $A_2(t)$  are shown in Figure 1. Then we convert the mortality matrix,  $q_{t,x}$  from (1) into a survival index,  $S_t$  for a cohort of age  $x$  at time  $t$ .

**Figure 1** The estimated parameters in the CBD model for men aged 60-84 from 1965 to 2003.



There are three types of liabilities in our numerical examples: whole-life annuity, whole-life insurance, and 20-year term-life insurance. The whole-life annuity is issued for men aged 60 and the cohort groups are paid \$1 at the end of each year. The whole-life insurance is issued for men aged 60 or 40, and the payout benefit is \$100. The term-life insurance is issued for men aged 40 and the payout benefit is also \$100. Both premiums are collected in a single premium today. The deferred periods are zero for simplicity. The interest rate is 3%, and the mortality process follows the CBD two-factor model. The liabilities’ expected values are \$14.94, \$74.72/\$54.41, and \$29.76, respectively. The information is summarized in Table 1. We calculate the expected values of liabilities on the basis of the mortality distributions generated by JPMorgan LifeMetrics (2006).<sup>3</sup>

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<sup>3</sup>If the market prices of mortality-linked securities are available, the mortality distribution could be transformed into a pricing distribution. For more details, see Cairns et al. (2006b).

Table 1 Basic assumptions about liabilities

	Whole-life annuity	Whole-life insurance	Term-life insurance
Age/Gender	60/Men	60 or 40/Men	40/Men
Payout Benefit	\$1 per year	\$100	\$100
Coverage	whole life	whole life	20 years
Premium Type	single	single	single
Interest Rate	3%	3%	3%
Deferred Period	immediately	immediately	immediately
Mortality Process	CBD	CBD	CBD
Premium Value	\$14.94	\$74.72/\$54.41	\$29.76

## 4 Numerical Analysis for the Optimal Liability Allocation

To demonstrate the hedging effect, we build three examples in two scenarios. The first scenario is when the insurance companies only care about risk reduction and do not consider the required profit loading. We choose a two-liability framework and compare the hedging results of the CVM and MDM approaches. We show that the CVM approach has a better hedging effect in terms of distribution than the MDM approach. Then we extend the analysis to the multi-liability framework in Scenario 2. We provide a three-liability example with the required loading constraint and find the optimal allocation. The results show that the CVM approach achieves a good hedging effect under the required profit constraint.

In the simulation, we first generate 10,000-times mortality processes to obtain the distributions of  $v^i$ . The loss distribution,  $r^i$ , is the liability value minus its expected value, divided by the expected value as shown in equation (8). With these returns, we estimate the mix weights in  $w^D$  (duration match),  $w^C$  (convexity adjustment) and  $w^{\text{CVM}}$  (CVM). The confidence intervals are chosen as  $\alpha = 99\%$ ,  $95\%$  and  $90\%$ . To implement the CVM optimization, we follow the methodology of Rockafellar and Uryasev (2000). The algorithm is implemented in C++ and we used the CPLEX 7.0 Callable Library to solve the linear programming problem.

### 4.1 Scenario 1: Pure Risk Reduction and Two Liabilities Hedging

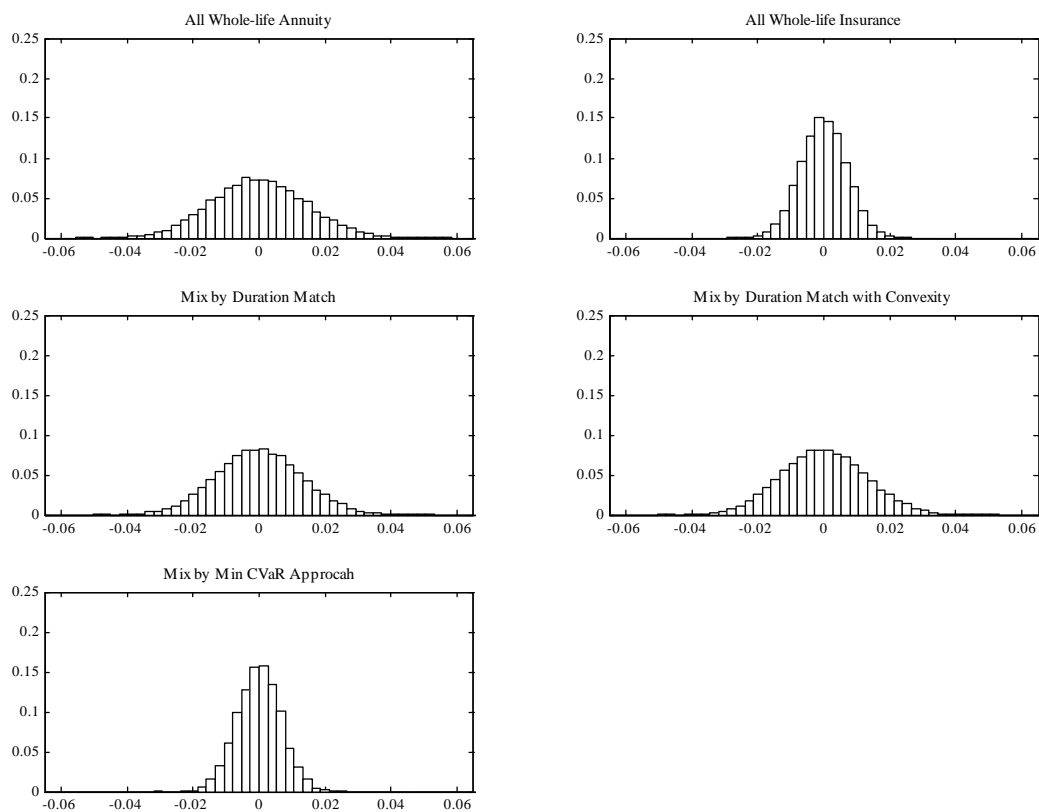
We consider a two-liability framework in this scenario. There are five cases. Case 1 is the distribution of the whole-life annuity; Case 2 is whole-life insurance; Case 3 is the mixed distribution of  $w^D$ ; Case 4 is the mixed distribution of  $w^C$  and Case 5 is the mixed distribution of  $w_\alpha^{\text{CVM}}$ . We solve  $w_\alpha^{\text{CVM}}$  by the following approach:

$$\begin{aligned}
 \mathbf{Min}_w \quad & E[r_p | r_p \geq r_p(\alpha)] \\
 \text{s.t.} \quad & w^a + w^l = 1, \text{ and } 0 \leq w^a, w^l \leq 1.
 \end{aligned} \tag{13}$$

where  $w^a$  and  $w^l$  are the weights for the annuity and life insurance, respectively.

The loss distributions are shown in Figure 2. In Figure 2, all annuity (Case 1) has the widest distribution among all cases and it represents a high-risk product. The distribution of all life insurance (Case 2) has a more central distribution and lower risk than Case 1. The annuity issued for men aged 60 has a wider distribution than the life insurance issued for men aged 40 before hedging. We find a narrower distribution in Case 3, in which the annuity distribution is mixed with some life insurance,  $w^D=10.6\%$ , as suggested by the MDM approach. We have a narrower and centered loss distribution in Case 4, but the effect of the convexity adjustment does not cause a significant improvement in the tail distributions. The postulation is that the convexity adjustment does not work well with distribution risk. The risk is significantly reduced in Case 5; the distribution reveals higher frequency in the center and lower frequency in the tails after hedging under the weight,  $w_{99\%}^{CVM}=67.3\%$ . The hedging result for the CVM approach has the smallest tail risk among Cases 3-5.

**Figure 2** The loss distributions for whole-life annuities at age 60 and whole-life insurance at age 40 (x-axis: values of  $r_p$ , y-axis: relative probability)



**Table 2** The CVaRs and statistics of the loss distributions for Figure 2

Model	$\alpha = 0.99$	$\alpha = 0.95$	$\alpha = 0.90$	Average	Std.* $10^2$ .	$w^l * 10^2$	Loading* $10^4$
All whole-life annuity	3.712	0.580	0.247	0.000	1.428	0.0	1.438
All whole-life insurance	1.853	0.286	0.122	0.000	0.696	100.0	0.192
Duration match	3.324	0.519	0.222	-0.000	1.279	10.6	1.306
Duration with convexity	3.348	0.523	0.223	0.000	1.288	9.9	1.314
The CVM approach	1.725	0.270	0.116	0.000	0.662	67.3	0.599

\*The Std. and  $w^l$  are multiplied by 100. The loading values are multiplied by 10,000, for example, 1.438 is 0.0001438.

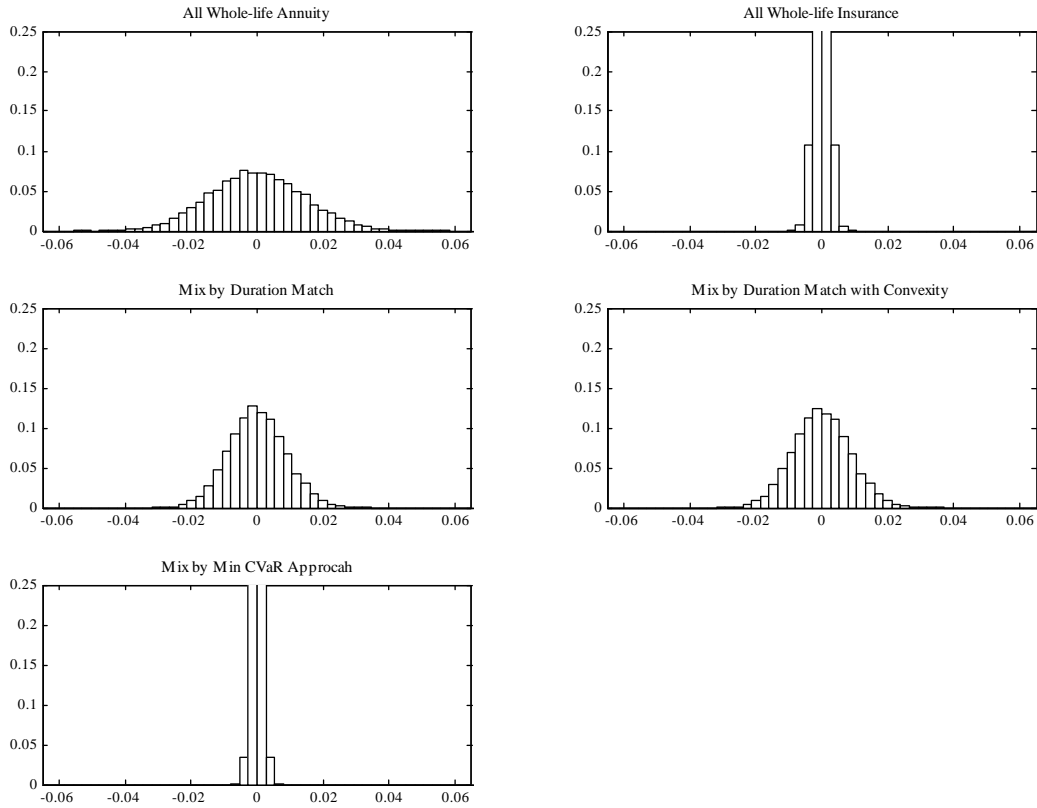
The CVaRs and statistics are shown in Table 2. The CVaRs of annuity and life insurance are 3.712 and 1.853, respectively. Hedged by the MDM and MDM with convexity, the 99% CVaRs decrease to 3.324 and 3.348. The CVM approach decreases the 99% CVaR to 1.725. We can see similar effects for other confidence intervals, i.e. 95% and 90%. The CVM approach offers the smallest CVaR, which is even smaller than that for whole life insurance. Column 5 shows that the CVM also has the smallest standard deviations (0.662%). This approach achieves a hedging effect in terms of the variance, too. We do not add the required loading constraint in this scenario, but just present the weighted loadings in the last column. The CVM approach has a lower weighted loading profit (0.599).

We replace the whole-life insurance for the age-60 cohort with the age-40 cohort in Figure 3. In Figure 3, the distribution for the whole-life insurance has a more centered distribution than the one in Figure 2. To reflect this change, the MDM approach holds more life insurance and increases  $w^D$  from 10.6% to 36.7% (see Column 6 of Table 3). However, the CVM approach increases the weight  $w_{99\%}^{CVM}$  from 67.3% to 87.2%. The CVM approach recommends much more life insurance than the MDM approach.

Table 3 shows the CVaRs of these distributions. If we take the 99% CVaR in Column 1 as an example, the whole-life annuity and whole-life insurance CVaR values are 3.712 and 0.583, respectively. After being hedged by the MDM and MDM with the convexity, the CVaRs decrease to 2.206 and 2.238, respectively. The CVM approach reduces CVaR to 0.379, the smallest value. These results show that the CVM approach offers a better hedging performance, in the scenario of the age-40 cohort.

In this risk reduction scenario, we tend to hold “too much” life insurance as the CVM approach suggests. However, the weighted profit loadings of the CVM approach are the smallest (0.222) as shown in the last column of Table 3. The loading constraints are included in the next scenario to fix this problem.

**Figure 3** The loss distributions for whole-life annuities and whole-life insurance both at age 60 (x-axis: values of  $r_p$ , y-axis: relative probability)



**Table 3** The CVaRs and statistics of the loss distributions for Figure 3

Model	$\alpha = 0.99$	$\alpha = 0.95$	$\alpha = 0.90$	Average	Std.* $10^2$ .	$w^l*10^2$	Loading* $10^4$
All whole-life annuity	3.712	0.580	0.247	0.000	1.428	0.0	1.438
All whole-life insurance	0.583	0.090	0.038	0.000	0.217	100.0	0.044
Duration match	2.206	0.345	0.147	0.000	0.851	36.7	0.926
Duration with convexity	2.238	0.350	0.149	0.000	0.863	35.9	0.938
The CVM approach	0.379	0.059	0.024	0.000	0.147	87.2	0.222

\*The Std. and  $w^l$  are multiplied by 100. The loading values are multiplied by 10,000, for example, 1.438 is 0.0001438.

## 4.2 Scenario 2: Multi-Liability Allocation with Required Loading

Assume that the insurance company sells three life insurance products in the market. The three liabilities are whole-life annuities for the age-60 cohort, whole-life insurance and 20-year term-life insurance for the age-40 cohort. The CVM approach with the required loading constraint

is

$$\mathbf{Min}_w \quad E[r_p | r_p \geq r_p(\alpha)] \quad (14)$$

$$s.t. \quad w^a \cdot \pi^a + w^{l1} \cdot \pi^{l1} + w^{l2} \cdot \pi^{l2} \geq \bar{\pi} \quad (15)$$

$$w^a + w^{l1} + w^{l2} = 1, \text{ and } 0 \leq w^a, w^{l1}, w^{l2} \leq 1. \quad (16)$$

where  $w^a$  is the weight of the whole-life annuity,  $w^{l1}$  is the weight of the whole-life insurance, and  $w^{l2}$  is the weight of the term-life insurance.  $\pi^a$ ,  $\pi^{l1}$ , and  $\pi^{l2}$  are profit loadings on the annuity, whole-life and term-life insurance, respectively, and are 1.438, 0.192, and 1.268 basis points, as implied by the premium pricing principle with the Sharpe ratio being equal to 15%. Taking the whole-life insurance as an example,  $E(v^{l1})=54.41$ ,  $\sigma(r^{l1})=0.00696$  (in Table 4), and  $SR=15\%$  indicates that  $\pi^{l1}=0.192$  basis points.<sup>4</sup> These loadings reflect the profit requested by the shareholders bearing the mortality systematic risk in the capital market. The higher the standard deviation, the higher the profit loading. The target return,  $\bar{\pi}$ , is set equal to 1 basis point. The CVaRs of the three liability distributions and their product mix are shown in Table 4.

In Table 4, all whole-life insurance has the small CVaR value, 1.853 under  $\alpha = 0.99$ . However, the CVM approach can not hold this liability too much under the inequality constraint. The CVM approach proceeds with the trade-off between  $r_p$  and  $\pi^i$ , and then gives  $w^a = 50.6\%$ ,  $w^{l1} = 27.6\%$ , and  $w^{l2} = 21.8\%$  under the 99% confidence interval. In row 4 of Table 4, we have mild CVaRs, for example, CVaR=2.666 under  $\alpha = 0.99$ . The weighted loading is 1.056 basis points, which is very close to 1 basis point and means that the constraint is active. The other two mortality confidence intervals, 95% and 90%, lead to similar hedging results.

**Table 4** The CVaRs and statistics of the loss distributions with  $\bar{\pi}=1$  bps

Model	$\alpha = 0.99$	$\alpha = 0.95$	$\alpha = 0.90$	Average	Std.* $10^2$ .	$w^a$	$w^{l1}$	$w^{l2}$	Loading* $10^4$
All whole-life annuity	3.712	0.580	0.247	0.000	1.428	100.0	0.0	0.0	1.438
All whole-life insurance	1.853	0.286	0.122	0.000	0.696	0.0	100.0	0.0	0.192
All term-life insurance	6.532	1.017	0.435	0.000	2.517	0.0	0.0	100.0	1.268
The CVM approach	2.666	0.422	0.181	0.000	1.059	50.6	27.6	21.8	1.056

\*The Std. and  $w^l$  are multiplied by 100. The loading values are multiplied by 10,000, for example, 1.438 is 0.0001438.

## 5 Conclusion and Discussion

In this article, we propose a new approach to optimize the allocation of insurer's liabilities under systematic mortality risk. By incorporating the natural hedging strategy of Cox and Lin (2007), the two-factor stochastic mortality model of Cairns et al. (2006b), and the Sharpe ratio loading price of Milevsky et al. (2006), we construct a CVM approach to evaluate the liability

<sup>4</sup>Here we assume that the premium loadings are given. That is, the firm with a natural hedging strategy can take a free ride on others without natural hedging.

allocations. We consider two numerical scenarios: the two-liability case without a loading constraint and the multi-liability case with a loading constraint. In the first scenario, the CVM approach exerts a better risk-reduction effect than the MDM approach. In the second scenario, the three-liability example reveals a trade-off effect between the CVaR and the required loadings. The results show that the proposed CVM approach leads to an optimal liability allocation and effectively reduces the mortality risks associated with forecasting longevity patterns for life insurance companies.

Some important issues for future research and practice clearly deserve more investigation. First, we deal with the parameter risk, but ignore the misspecification or modeling risk. For example, the real mortality process may not follow the CBD model. Second, we omit the basis risk of the mortality rate between life insurance and annuities because of the data limitations. In our numerical example, we assume the mortality processes for life insurance and annuities are the same. In fact, the mortality experiences may differ for these products. Third, the premium loadings for each liability are decided individually by means of the Sharpe ratio in this study. To maintain rigidity, they should be priced according to their contributions to the aggregated risk, in a similar way to the “beta” concept of the Capital Asset Pricing Model (CAPM). This work is beyond the scope of this paper and so we leave it for future study. Finally, we illustrate the hedging strategy with a mortality term structure, but a flat interest rate yield curve. An analysis of the combined effects of stochastic mortality and stochastic interest rate would offer more realistic results.

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