

On the Determination of Capital Charges in a Discounted Cash Flow Model

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Abstract

We derive formulas for calculating the premiums that should be charged on policies in a discounted cash flow model with tax reserves and required assets that are determined by regulation. We also determine the unique division of required assets into “reserves” and “capital” that allows the product profitability to be correctly evaluated. That is, the profit after capital charges is zero if the product achieves the return assumed in pricing. We illustrate the concepts using whole life insurance and guaranteed minimum death benefit examples.

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1. Introduction

The importance of economic capital to insurance companies has been growing in recent years. To some extent, this has been driven by regulation, as illustrated by the introduction of the Solvency II capital standards in Europe. On the other hand, it is being increasingly seen as good business practice independent of any regulatory action. The implementation of an economic capital framework raises a number of practical questions, such as what solvency threshold to use, whether to use value-at-risk or a coherent risk measure and how to allocate the capital to lines of business. Both the determination of the capital level and the allocation of capital to individual lines of business are important for the evaluation of products and product managers. Typically, the income of the line is computed and the allocated capital is charged its cost and subtracted. A positive result indicates a profitable line. In a market-value driven Economic Capital framework such as Solvency II in Europe, it is actually not capital per se but an asset level that is determined and allocated. This paper addresses this issue of dividing the allocated assets into “capital” and “reserves” for the purposes of accurately evaluating the profitability of a product.

It may seem at first glance that this question is trivial. Insurance companies already routinely divide assets this way and often determine capital charges to the lines based on the hurdle rate. On closer inspection, however, the appropriate reserve to use is not obvious. There are at least 6 candidates for reserves:

1. U.S. Statutory Reserves;
2. U.S. GAAP Reserves;

3. U.S. Tax Reserves;
4. Fair Value of Liabilities;
5. Assets at a somewhat conservative solvency standard (Solvency II uses 75%);
6. Expected Loss under the realistic measure discounted at the risk-free rate.

Each of these reserve bases has a corresponding capital level. The income less capital charges will be different for all six candidates both because the change in reserves is different and the capital level being charged (i.e. assets less reserves) is different.

Therefore, each of the six corresponding profitability results will evaluate the line and manager differently. Before reading further, I would suggest that the reader decide whether one of these six capital amounts is the correct one for assessing capital charges when evaluating product profitability, or whether the correct answer is a seventh possibility.

The question of which definition is appropriate for evaluation purposes can be answered in the context of discounted cash flow analysis of Insurance Company profits. The method was first analyzed by Myers and Cohn (1987) and later by Cummins (1990) and Taylor (1994). A good overview of the current state can be found in Cummins and Phillips (2000). While most of the analysis has been done in a property-casualty insurance framework, the papers include multi-period models and are therefore applicable to life insurance as well.

In these models, it is assumed that the insurance company receives and pays cash flows every period. In addition, the company is required to hold assets every period in order to maintain a solvency standard. The interest on these assets is added to the product cash flows, and the combined flows are assumed to be paid out to shareholders net of

funds needed to increase or decrease the assets held in the next period. The discount rate for these cash flows must then be determined.

There are several prominent candidates for the discount rate “ x ”. We will see in the next sections that this rate should be equal to the hurdle rate used in pricing. The CAPM implies that x should equal the risk free rate plus a term related to the covariance of the cash flow with the market, that is $x = r_f + \beta(r_m - r_f)$. Froot and Stein (1998) suggest a hurdle rate of $x = r_f + \beta(r_m - r_f) + G \text{cov}(\varepsilon_N^N, \varepsilon_P^N)$ where G is a risk-aversion coefficient for the company, and ε_N^N and ε_P^N are the non-tradeable portions of the new risk and the existing portfolio risk.

Alternatively, the rate “ x ” could be viewed as the one that produces the premiums for the risk being examined. The product could be viewed as a complicated derivative in which case the premium would be determined by discounting the risk-neutral cash flows at the risk-free rate, or the actual observed market premiums could be used. Regardless of the manner in which “ x ” is chosen, however, this rate will also be identical to the rate charged as the cost of capital.

Once this rate is determined, there will be several deviations from the approach taken in Taylor. He assumes that the reserves for tax purposes and the reserves that are used for the determination of capital are identical and equal to the “technical reserves”, that is the discounted value of the expected losses. In this model, the only free parameter is the “capital”. Taylor finds that this capital must roll off approximately proportionally to the reserves over time in order to achieve the desired rate of return. In reality, however, the tax reserves may not be equal to the technical reserves. More importantly, the “capital” is not usually a free parameter, but the total assets will be constrained by the

solvency standard. In this paper, we view the division of assets between “capital” and “reserves” to be the free parameter for the determination of capital charges.

The paper is organized as follows: In section 2, we describe the problem, define notation and derive some general formulas. In section 3, we apply these formulas to the specific case of a Solvency II-like asset standard. In section 4, we illustrate the use of the formulas for some specific life insurance products. Section 5 contains the conclusions.

2. Notation and Description of the Problem

2.1 Single Premium and Single Loss

We begin by assuming a premium P is charged at time 0 to cover a loss at integral time T . Taxes at rate τ will be paid on both investment and underwriting profits at the end of every integral time period. In addition, a tax reserve $V_{i,T}^{(\tau)}$ is held at every integral time $1 \leq i \leq T - 1$. The tax reserve is determined by regulation. It may depend on the information known at that time, i.e. the “filtration” \mathfrak{F}_{i-1} . Assets $A_{i,T}$ are held at every integral time $0 \leq i \leq T - 1$. Assets are also assumed to be fixed as they are determined according to a regulatory or internal capital standard. It is further assumed that the investment should earn $x\%$ in order to be profitable. A capital charge of $x\%$ will be assessed against the capital held to support the loss. A positive profit net of capital charges will indicate a profitable product and a negative profit net of capital charges will indicate an unprofitable product. If the premium is set to this standard, the mean loss should product zero profit net of capital charges.

At time 0, the insurer's cash flow is $CF_0 = P - A_{0,T}$. It is assumed that all taxes are paid at the end of the year, so at time 1, taxes are paid on the premium and tax reserves first appear. The total cash flow, including change in required capital, is $CF_1 = A_{0,T} - A_{1,T} - P\tau + A_{0,T}r_f(1-\tau) + V_{1,T}^{(\tau)}\tau$. At time $2 \leq i \leq T-1$, the cash flow is $CF_i = A_{i-1,T} - A_{i,T} + A_{i-1,T}r_f(1-\tau) + (V_{i,T}^{(\tau)} - V_{i-1,T}^{(\tau)})\tau$ and at time t , the cash flow is $CF_T = A_{T-1,T} - L_T(1-\tau) + A_{T-1,T}r_f(1-\tau) - V_{T-1,T}^{(\tau)}\tau$. Taking expected values and setting the discounted value at $x\%$ equal to zero leads to:

$$P = \frac{E[L_T]}{(1+y)(1+x)^{T-1}} + \sum_{i=0}^{T-1} \frac{A_{i,T}(y-r_f)}{(1+y)(1+x)^i} - \tau \sum_{i=1}^{T-1} V_{i,T}^{(\tau)} \frac{y}{(1+y)(1+x)^i} \quad (1)$$

where

$$y \equiv \frac{x}{1-\tau} \quad (2)$$

The rate y could be viewed as a before-tax hurdle rate. The first term in equation (1) represents the discounted value of the loss at the required earnings rate. The second term is zero if $y = r_f$ or $x = r_f(1-\tau)$. Therefore, this term represents a cost to holding the assets. The insurer wants to earn a return of x but can only earn $r_f(1-\tau)$ after tax. If what the insurer wants and what it can get are the same, this cost vanishes. The third term indicates that the premium can be reduced in the presence of tax reserves, as this shifts taxes paid (negative cash flows) to later periods.

An equivalent form of equation (1) with a different intuitive interpretation can be found by assuming $A_{i,T} = \frac{E[L_T]}{(1+r_f)^{T-i}} + \Delta A_{i,T}$ and $V_{i,T}^{(\tau)} = \frac{E[L_T]}{(1+r_f)^{T-i}} + \Delta V_{i,T}^{(\tau)}$. This gives:

$$P = \frac{E[L_T]}{(1+r_f)^T} + \sum_{i=0}^{T-1} \frac{\Delta A_{i,T}(y-r_f)}{(1+y)(1+x)^i} - \tau \sum_{i=1}^{T-1} \frac{\Delta V_{i,T}^{(\tau)} y}{(1+y)(1+x)^i} \quad (3)$$

In other words, a riskless loss with no taxes will have an actuarially fair premium. If the product is risky and compensation is required for this risk, the assets to be held will be larger than the actuarially fair reserve and the premium will increase. In the presence of taxes, tax reserves above the actuarially fair reserves will reduce the premiums. The derivation of equations (1) and (3) can be found in Appendix A.

Now, we turn our attention to the division between “Capital” and “Reserves” that accurately assesses whether the line earns its cost of capital. The assets required to be held have already been determined as $A_{i,T}$ from an outside solvency model. We need to define capital in such a way that if expectations are realized, the income less capital charges is equal to zero at rate $x\%$. We define $V_{i,t}^{(e)}$ to be the value of the reserve at time i that satisfies this requirement and will refer to it as the “evaluation reserve” since it will be used in the evaluation of product line profitability. The capital charges are taken as a percentage of the capital amount $A_{i,T} - V_{i,T}^{(e)}$. At time $T-1$, we have the following equation for this reserve:

$$-E[L_T](1-\tau) + A_{t-1,t} r_f (1-\tau) + V_{t-1,t}^{(e)} - V_{t-1,t}^{(\tau)} \tau - (A_{t-1,t} - V_{t-1,t}^{(e)})x = 0 \quad (4)$$

This takes into account the after-tax loss, after tax net investment income, the income from change in evaluation reserves, taxes paid on release of the tax reserve, and the capital charges. In general, the expected value in equation (4) should be conditional on the information \mathfrak{F}_{T-1} . In the current case, we will assume that the loss is not dependent on any information before time T and will, for simplicity, not include the

conditional expected value in the equations that follow. The filtration will matter,

however, in some of the examples in section 4. Equation (4) solves for:

$$V_{T-1,T}^{(e)} = \frac{E[L_T]}{(1+x)} + \frac{A_{T-1,T}[x - r_f(1-\tau)]}{1+x} - \tau \frac{(E[L_T] - V_{T-1,T}^{(\tau)})}{1+x} \quad (5)$$

For $0 < i < T-1$, equation (4) becomes:

$$A_{i,T}r_f(1-\tau) + (V_{i,T}^{(e)} - V_{i+1,T}^{(e)}) - (V_{i,T}^{(\tau)} - V_{i+1,T}^{(\tau)})\tau - (A_{i,T} - V_{i,T}^{(e)})x = 0 \quad (6)$$

It can be shown by induction that the reserve satisfies:

$$V_{i,T}^{(e)} = \frac{E[L_T]}{(1+x)^{T-i}} + \sum_{j=i}^{T-1} \frac{A_{j,T}[x - r_f(1-\tau)]}{(1+x)^{j-i+1}} - \tau \sum_{j=i}^{T-1} \frac{(V_{j+1,T}^{(\tau)} - V_{j,T}^{(\tau)})}{(1+x)^{j-i+1}} \quad (7)$$

with the definition $V_{T,T}^{(\tau)} = E[L_T]$. Finally, we find for $i = 0$:

$$V_{0,T}^{(e)} = P \quad (8)$$

being careful to note that equation (8) is valid just after the premium is paid, whereas

$V_{0,T}^{(e)} = 0$ just before the premium is paid. Now, using the same definitions as in equation

(3), we can rewrite equation (7) more intuitively as:

$$V_{i,T}^{(e)} = \frac{E[L_T]}{(1+r_f)^{T-i}} + \sum_{j=i}^{T-1} \frac{\Delta A_{j,T}[x - r_f(1-\tau)]}{(1+x)^{j-i+1}} - \tau \sum_{j=i}^{T-1} \frac{(\Delta V_{j+1,T}^{(\tau)} - \Delta V_{j,T}^{(\tau)})}{(1+x)^{j-i+1}} \quad (9)$$

The first term is the expected present value reserve. The second term is the present value of the difference between the hurdle rate and the after-tax rate you are capable of earning on excess assets. The third term is a reserve for the deviation of taxes from those that would be due if tax reserves were actuarially fair.

The derivation of equations (7)-(9) can be found in appendix B.

2.2 Multiple Premiums and Multiple Losses

The analysis in the case of multiple premiums and multiple losses is very similar to the single loss case if the assets needed at time i , A_i , are known at time zero, and are independent of the sequence of losses L_j that occur before time i . This will typically be the case if the losses are uncorrelated. If the losses are correlated and A_i is dependent on the losses prior to time i , A_i should be replaced by $E[A_i]$ in the premium equations below, and $V_i^{(e)}$ will be dependent on the sequence of prior losses with A_j replaced by $E[A_j | \mathfrak{F}_i]$. Similarly, the equations in this section assume premiums paid at the beginning of each period P_0, \dots, P_{T-1} are nonrandom and paid with certainty. Otherwise, the premiums should be replaced with their expected values given the available information. Following Appendix A for discounted cash flows, we get an equation analogous to Equation (1):

$$\sum_{i=0}^{T-1} \frac{P_i}{(1+x)^i} = \sum_{i=1}^T \frac{E[L_i]}{(1+y)(1+x)^{i-1}} + \sum_{i=0}^{T-1} \frac{A_i(y-r_f)}{(1+y)(1+x)^i} - \tau \sum_{i=1}^{T-1} V_i^{(\tau)} \frac{y}{(1+y)(1+x)^i} \quad (10)$$

If we assume $A_i = \sum_{j=i+1}^T \frac{E[L_j]}{(1+r_f)^{j-i}} - \sum_{j=i+1}^{T-1} \frac{P_j}{(1+r_f)^{j-i}} + \Delta A_i$ (the middle term is not present for

A_{T-1}) and $V_i^{(\tau)} = \sum_{j=i+1}^T \frac{E[L_j]}{(1+r_f)^{j-i}} - \sum_{j=i}^{T-1} \frac{P_j}{(1+r_f)^{j-i}} + \Delta V_i^{(\tau)}$ we find:

$$\sum_{i=0}^{T-1} \frac{P_i}{(1+r_f)^i} = \sum_{i=1}^T \frac{E[L_i]}{(1+r_f)^i} + \sum_{i=0}^{T-1} \frac{\Delta A_i(y-r_f)}{(1+y)(1+x)^i} - \tau \sum_{i=1}^{T-1} \frac{\Delta V_i^{(\tau)} y}{(1+y)(1+x)^i} \quad (11)$$

In other words, if required assets and tax reserves are their actuarially fair values, premiums will be actuarially fair. Extra assets are expensive and increase the premiums. Extra tax reserves are beneficial, and reduce the premiums. The derivation of equation (11) is found in Appendix C.

It is easiest to calculate premiums from Equation (11) when ΔA_i and $\Delta V_i^{(\tau)}$ do not depend on the values of P_i . This will almost never be the case, as the premiums and rates used in a tax reserve calculation are determined by regulation and fixed, and will only by coincidence be equal to the premiums determined from equation (11). In addition, the assets held under a given solvency standard will be different for different premiums, unless the tax reserve moves to exactly compensate, which it almost certainly will not in practice. Therefore, in most practical calculations, P_i variables will appear on the right hand side of equation (11), and more algebra will typically be necessary to reach a final answer.

We now calculate the evaluation Reserves can also be calculated. In the context of multiple premiums, Equation (6) becomes:

$$P_i(1-\tau) - E[L_{i+1}](1-\tau) + A_i r_f (1-\tau) + (V_i^{(e)} - V_{i+1}^{(e)}) - (V_i^{(\tau)} - V_{i+1}^{(\tau)})\tau - (A_i - V_i^{(e)} - P_i)x = 0 \quad (12)$$

assuming $V_i^{(e)}$ is defined to be the reserve just before the premiums are paid. This can be shown by induction to give:

$$V_i^{(e)} = \sum_{j=i+1}^T \frac{E[L_j]}{(1+x)^{j-i}} - \sum_{j=i}^{T-1} \frac{P_j}{(1+x)^{j-i}} + \sum_{j=i}^{T-1} \frac{A_j [x - r_f (1-\tau)]}{(1+x)^{j-i+1}} - \tau \sum_{j=i}^{T-1} \frac{[(V_{j+1}^{(\tau)} + E[L_{j+1}]) - [V_j^{(\tau)} + P_j]]}{(1+x)^{j-i+1}} \quad (13)$$

It should not be surprising that, in this context, Equation (9) becomes:

$$V_i^{(e)} = \sum_{j=i+1}^T \frac{E[L_j]}{(1+r_f)^{j-i}} - \sum_{j=i}^{T-1} \frac{P_j}{(1+r_f)^{j-i}}$$

$$+ \sum_{j=i}^{T-1} \frac{\Delta A_j [x - r_f (1 - \tau)]}{(1 + x)^{j-i+1}} - \tau \sum_{j=i}^{T-1} \frac{(\Delta V_{j+1}^{(\tau)} - \Delta V_j^{(\tau)})}{(1 + x)^{j-i+1}} \quad (14)$$

3. Some Solutions in a Solvency II context

3.1 Single Premium and Single Loss

In this case, we will assume that assets $A_{i,T}$ must be held to be 99.5% certain that the assets at time $i + 1$ will exceed the loss L_{i+1} plus the market value at time $i + 1$ of the losses L_j , $i + 1 < j < t$. The one period case is the simplest to analyze, and we will then generalize to the multiperiod case. In one year, the market value of assets is

$A_{0,1}[1 + r_f(1 - \tau)] - L_1(1 - \tau) - P\tau$ and market value of liabilities is 0. We want

$$\Pr[A_{0,1}[1 + r_f(1 - \tau)] - L_1(1 - \tau) - P\tau \geq 0] = 0.995 \quad (15)$$

or

$$\Pr\left[L_1 \leq A_{0,1} \left(\frac{1}{(1 - \tau)} + r_f \right) - \frac{P\tau}{(1 - \tau)}\right] = 0.995 \quad (16)$$

This means

$$A_{0,1} = \frac{\xi_{L_1}^{0.995} (1 - \tau) + P\tau}{[1 + r_f(1 - \tau)]} \quad (17)$$

where $\xi_{L_1}^{0.995}$ is the 99.5th percentile of L_1 . Given these starting assets, the premium is

found to be:

$$P = \frac{E[L_1] + (\xi_{L_1}^{0.995} - E[L_1])R}{(1 + r_f)} \quad (18)$$

where

$$R \equiv \frac{x - r_f(1 - \tau)}{1 + x} \quad (19)$$

R is the “cost” of additional safety beyond the mean loss. Note that the combination $E[L_1] + (\xi_{L_1}^{0.995} - E[L_1])R$ will occur frequently in the following analysis.

Now let’s consider the multiperiod case. We’ll first look at the last period to determine $A_{T-1,T}$. The market value of assets at time T are

$A_{T-1,T}[1 + r_f(1 - \tau)] - L_T(1 - \tau) - V_{T-1,T}^{(\tau)}\tau$ and the market value of liabilities is 0. This gives

$$A_{T-1,T} = \frac{\xi_{L_T}^{0.995}(1 - \tau) + V_{T-1,T}^{(\tau)}\tau}{[1 + r_f(1 - \tau)]} \quad (20)$$

In order to determine $A_{i,T}$ at earlier times, it will be necessary to determine the market value of L_T at time i . If we define $\tilde{P}_{i,T}$ to be the premium that would be charged at time i to cover a loss at time t , this must also be the market value of that loss. Moreover, we have $MV_0[L_{[T-i]}] = MV_i[L_T]$, where $MV_0[L_{[T-i]}]$ is defined to be the market value at time 0 of a loss with the same distribution as L_T but occurring at time $T - i$. Appendix D gives an arbitrage argument that shows it is true for $i = T - 1$.

Equation (1) allows us to determine the premium recursively. If we know

$MV_0[L_{[T-j-1]}] = MV_{j+1}[L_T]$ we can find $MV_0[L_{[T-j]}] = MV_j[L_T]$ from

$$MV_j[L_T] = \frac{MV_{j+1}[L_T]}{(1 + r_f)}(1 + r_\tau) - r_\tau \frac{V_{j+1,T}^{(\tau)}}{(1 + r_f)} \quad (21)$$

with

$$r_\tau \equiv \frac{\tau x}{(1 - \tau)(1 + x)} = \frac{1 + y}{1 + x} - 1 \quad (22)$$

which gives

$$P = MV_0[L_T] = \left[\frac{E[L_T] + (\xi_{L_t}^{0.995} - E[L_T])R}{(1+r_f)^T} \right] (1+r_\tau)^{T-1} - r_\tau \sum_{i=1}^{T-1} \frac{V_{i,T}^{(\tau)}}{(1+r_f)^i} (1+r_\tau)^{T-1} \quad (23)$$

The recursion in Equation (21) can also be written as

$$MV_j[L_t] = \frac{MV_{j+1}[L_T]}{(1+r_f)} + r_\tau \left[\frac{MV_{j+1}[L_T] - V_{j+1,t}^{(\tau)}}{(1+r_f)} \right] \quad (24)$$

which implies that if the tax reserves are equal to the market value or the tax rate is zero, then the market value is discounted period-by-period at the risk-free rate. The details are in Appendix E.

Given the market values in Equation (23), we can now find the assets $A_{i,T}$ for $0 < i < T-1$. At time $i+1$, the market value of assets is

$A_{i,T}[1+r_f(1-\tau)] + (V_{i+1,t}^{(\tau)} - V_{i,t}^{(\tau)})\tau$ and the market value of liabilities is

$$\left[\frac{E[L_T] + (\xi_{L_t}^{0.995} - E[L_T])R}{(1+r_f)^{T-i-1}} \right] (1+r_\tau)^{T-i-2} - r_\tau \sum_{j=i+2}^{T-1} \frac{V_{j,T}^{(\tau)}}{(1+r_f)^{j-i-1}} (1+r_\tau)^{j-i-2}, \text{ and we want:}$$

$$\Pr \left[A_{i,T}[1+r_f(1-\tau)] + (V_{i+1,T}^{(\tau)} - V_{i,T}^{(\tau)})\tau - \left\{ \left[\frac{E[L_T] + (\xi_{L_t}^{0.995} - E[L_T])R}{(1+r_f)^{T-i-1}} \right] (1+r_\tau)^{T-i-2} - r_\tau \sum_{j=i+2}^{T-1} \frac{V_{j,T}^{(\tau)}}{(1+r_f)^{j-i-1}} (1+r_\tau)^{j-i-2} \right\} \geq 0 \right] = 0.995 \quad (25)$$

The left hand side in Equation (25) is a certain quantity, so we find

$$A_{i,t} = \frac{\left[\frac{E[L_T] + (\xi_{L_t}^{0.995} - E[L_T])R}{(1+r_f)^{T-i-1}} \right]}{[1+r_f(1-\tau)]} (1+r_\tau)^{T-i-2} - \frac{(V_{i+1,T}^{(\tau)} - V_{i,T}^{(\tau)})\tau}{[1+r_f(1-\tau)]}$$

$$\frac{-r_\tau \sum_{j=i+2}^{T-1} \frac{V_{j,T}^{(\tau)}}{(1+r_f)^{j-i-1}} (1+r_\tau)^{j-i-2}}{[1+r_f(1-\tau)]} \quad (26)$$

Substituting Equation (26) into Equation (7) and after some algebra we find the following equation for the evaluation reserve:

$$\begin{aligned} V_{i,T}^{(e)} = & \frac{E[L_T]}{(1+x)^{T-i-2} [1+r_f(1-\tau)](1+r_f)} \left[1 + \frac{r_f \tau}{1+x} + r_\tau R + (1+r_\tau) R \ddot{s}_{\overline{T-i-2}|r_{ann}} \right] \\ & + \frac{(\xi_{L_t}^{0.995} - E[L_T])R}{(1+x)^{T-i-2} [1+r_f(1-\tau)](1+r_f)} \left[1 - \frac{\tau}{1+x} + R \ddot{s}_{\overline{T-i-2}|r_{ann}} \right] \\ & - \sum_{j=i}^{T-1} \frac{(V_{i+1,T}^{(\tau)} - V_{i,T}^{(\tau)})\tau}{[1+r_f(1-\tau)](1+x)^{j-i}} - r_\tau \sum_{j=i+2}^{T-1} \frac{V_{j,T}^{(\tau)} R \ddot{s}_{\overline{j-i-1}|r_{ann}}}{(1+x)^{j-i-1} [1+r_f(1-\tau)](1+r_\tau)} \end{aligned} \quad (27)$$

with all accumulation symbols being computed at rate:

$$r_{ann} = \frac{(1+r_\tau)(1+x)}{(1+r_f)} - 1 = \frac{1+y}{1+r_f} - 1 \quad (28)$$

The equations are linear if the total loss consists of uncorrelated losses at different times.

We have now reached the point of diminishing returns in determining exact formulas. The next section will work out some illustrative and practical examples.

4. Examples in a Solvency II context

4.1 A Single Loss with a Single Premium

Equations (3) and (9) carry the most intuitive information on premiums and evaluation reserves. On the other hand, Equations (1) and (7) are the most useful for practical calculations. We will begin with a simple illustrative example where the loss

insured has an expected value of 500, a 99.5th percentile of 700 and occurs five years after issue. We will assume that the risk-free rate $r_f = 6%$, the required return $x = 10%$ and the tax rate $\tau = 34%$. Tax reserves are calculated as determined by the taxing authority, which we will assume has mandated that they be the expected value of the loss discounted to the valuation date at a valuation rate of 7%. The premium is calculated to be 385.1821286. Table I shows the Tax Reserves and Required Assets, both in absolute terms and the deviations from Expected Present Value for use in Equations (3) and (9). It also includes Evaluation Reserves, Capital and Market Values. Table II shows the cash flows at each time including change in assets held. It also shows the three pieces of Equation (4). Namely, the income in each period not including change in evaluation reserves, the change in evaluation reserves and the capital charges at 10% if the expected loss is the actual loss. The cash flows have a present value of zero at 10%, and the income plus change in evaluation reserves plus capital charges are zero at all times.

4.2 A Simple Two Period Loss Model

We now turn to a model with two uncorrelated losses funded by two equal premiums defined in the contract. $E[L_1] = 400$, $\xi_{L_1}^{0.995} = 500$, $E[L_2] = 500$ and $\xi_{L_2}^{0.995} = 700$. Again, the risk-free rate $r_f = 6%$, the required return $x = 10%$ and the tax rate is $\tau = 34%$. Tax reserves are calculated as equivalence principle reserves at 7% as of the valuation date. The simplest method to find the premium is to work backwards from time 2. Make a guess for premiums $P_0 = P_1 = \Pi$. Then, find the assets from:

$$A_i = \frac{\xi_{L_{i+1}}^{0.995} (1 - \tau) + (V_i^{(\tau)} - V_{i+1}^{(\tau)})\tau + \tau P_i + MVL_{i+1}}{[1 + r_f(1 - \tau)]} \quad (29)$$

and the market value of liabilities from:

$$\begin{aligned}
MVL_i = & \sum_{j=i+1}^T \frac{E[L_j]}{(1+y)(1+x)^{j-i-1}} + \sum_{j=i}^{T-1} \frac{A_j(y-r_f)}{(1+y)(1+x)^{j-i}} \\
& - \tau \sum_{j=i+1}^{T-1} V_j^{(\tau)} \frac{y}{(1+y)(1+x)^{j-i}} - \sum_{j=i}^{T-1} \frac{P_j}{(1+x)^{j-i}}
\end{aligned} \tag{30}$$

then solve for the value of $P_0 = P_1 = \Pi$ that sets $MVL_0 = 0$. Given the assumptions above, we find $\Pi = 430.9106895$.

Table III shows the Tax Reserves and Required Assets, both in absolute terms and the deviations from Expected Present Value for use in Equations (11) and (14). It also includes Evaluation Reserves, Capital and Market Values. Table IV shows the cash flows at each time including change in assets held. It also shows the income in each period not including change in evaluation reserves, the change in evaluation reserves and the capital charges at 10% if the expected loss is the actual loss. The cash flows have a present value of zero at 10%, and the income plus change in evaluation reserves plus capital charges are zero at all times, as in the previous example.

4.3 Two Year Term Life Insurance

We now reach our first real world example with correlated losses. We will model a two-year term insurance with 100,000 face amount. There are 1000 identical independent individuals with $q_x = 0.020$ and $q_{x+1} = 0.025$. We will view N_i as a random variable representing the number alive at time i and D_i as a random variable representing the number of deaths between time i and $i+1$. In order to solve Equation (10) for the premium, we need A_0 , $E[A_1]$ and $E[P_1]$. Assume we charge a premium Π

per individual alive at the beginning of the period. $P_0 = 1000\Pi$ and $E[P_1] = 980\Pi$. Tax reserves are assumed to be zero. The required assets at time 1 viewed as a function of the number alive at time 1 are given by:

$$A_1(N_1) = \frac{\xi_{L_2|N_1}^{0.995}(1-\tau) + \pi N_1 \Pi}{[1+r_f(1-\tau)]} \quad (31)$$

$L_2 | N_1$ is 100,000 times a binomial variable with N_1 trials and probability q_{x+1} . The 99.5th percentile can be determined from this information, and so can $A_1(N_1)$. N_1 is a binomial variable with 1000 trials and probability $p_x = 1 - q_x$. Using this information, we find that $E[A_1] = 2,412,312.05 + 320.5078876\Pi$.

To find A_0 , we need to be sure that assets are 99.5% certain to exceed market value of liabilities at time 1. Both assets and market value of liabilities will be dependent on N_1 . Assets are given by:

$$MVA_1 = A_0[1+r_f(1-\tau)] - 1000\Pi\tau - 100,000*(1000 - N_1)(1-\tau) \quad (32)$$

and liabilities by:

$$MVL_1 = \frac{E[L_2 | N_1]}{(1+y)} - N_1\Pi + \frac{A_1[N_1](y-r_f)}{(1+y)} \quad (33)$$

and we know $E[L_2 | N_1] = 100,000q_{x+1}N_1$. Since N_1 is binomial, the 99.5% solvency criterion involves setting $MVA_1 = MVL_1$ at $N_1 = 968$. This solves for

$$A_0 = 4,237,501.48 - 579.87683598\Pi. \text{ Now we can determine } \Pi = 2,185.20.$$

We can now use Equation (13) to find the expected evaluation reserve $E[V_1^{(e)}] = 161,338.82$. Ultimately, $V_1^{(e)}$ is a function of N_1 since several pieces of Equation (13) are functions of N_1 . Tables V and VI show balance sheet and income

statement items assuming that the realized values of random variables are their expected values. This is not quite the same as saying the expected number of people die, since $E[A_1(N_1)] \neq A_1(E[N_1])$ because A_1 is a nonlinear function of N_1 . For instance, $E[A_1(N_1)] = 3,112,684.37$ while $A_1(980) = 3,112,838.65$.

4.4 Whole Life Insurance

We now model whole life insurance. The insurance has been issued to 1000 people age 40 with a face amount of 100,000. We will assume mortality follows the 1980 CSO table, which has a maximum age of 100. We will therefore work backwards from this age. We define N_i , $0 \leq i \leq T$ to be a random variable representing the number of the 1000 starting individuals still alive at time i . D_i is a random variable representing the number who die between time i and time $i+1$. A premium Π per person must be paid at the beginning of each period. To compute Π , we must compute the following five quantities working backward from age 100.

First, we need $\xi_{D_i|N_i}^{0.995}$ so that we can compute the amount of assets needed at all combinations of i and N_i to meet the Solvency II safety requirements. We are assuming that binomial stochastic mortality is the only risk. This is obviously a simplification of the real world situation, but this assumption is useful for demonstration purposes. Adding additional risks will change the answer obtained, but not the method of solution except in the case of stochastic interest rates, which is beyond the scope of this paper. $\xi_{D_i|N_i}^{0.995}$ can be computed from a binomial distribution with N_i trials with a probability of “success” of q_{x+i} .

Second, we need $\Pr[N_j | N_i]$ for $j \geq i$. This can be computed from the binomial distribution with N_i trials and probability p_{x+i} .

Third, we can find $A_i[N_i]$, the required assets at time i if N_i individuals are alive, if we already know the values of $MVL_{i+1}[N_{i+1}] = c_{i+1}[N_{i+1}] + d_{i+1}[N_{i+1}]\Pi$, the market value of liabilities at time $i+1$ if N_{i+1} individuals are alive. In order to guarantee market value of assets are greater than market value of liabilities 99.5% of the time, we need:

$$A_i[N_i] = \frac{100,000 \xi_{D_i|N_i}^{0.995} (1-\tau) - \tau \left[(N_i - \xi_{D_i|N_i}^{0.995}) \tilde{V}_{i+1}^{(\tau)} - N_i \tilde{V}_i^{(\tau)} \right] + c_{i+1} [N_i - \xi_{D_i|N_i}^{0.995}]}{[1 + r_f (1 - \tau)]} + \frac{\tau N_i + c_{i+1} [N_i - \xi_{D_i|N_i}^{0.995}]}{[1 + r_f (1 - \tau)]} \Pi = a_i [N_i] + b_i [N_i] \Pi \quad (34)$$

$\tilde{V}_i^{(\tau)}$ is the tax reserve per individual alive at time i .

Fourth, we can now find $E_k [A_i [N_k]]$, which is the expected value at time $0 \leq k \leq i$ of the assets needed at time i given that there are N_k individuals alive at time k as:

$$E_k [A_i [N_k]] = \sum_{N_i=0}^{1000} \Pr[N_i | N_k] A_i [N_i] = e_{ki} [N_k] + f_{ki} [N_k] \Pi \quad (35)$$

Finally, we can find the market value of liabilities at time i for each N_i as:

$$MVL_i [N_i] = \sum_{j=i+1}^t \frac{E[L_j | N_i]}{(1+y)(1+x)^{j-i-1}} + \sum_{j=i}^{t-1} \frac{E_i [A_j [N_i]] (y - r_f)}{(1+y)(1+x)^{j-i}} - \tau \sum_{j=i}^{t-1} \frac{y E_i [N_j \tilde{V}_j^{(\tau)}]}{(1+y)(1+x)^{j-i}} - \sum_{j=i}^{t-1} \frac{E_i [N_j \Pi]}{(1+x)^{j-i}} \quad (36)$$

which becomes:

$$\begin{aligned}
MVL_i[N_i] &= \left\{ \sum_{j=i+1}^t \frac{100,000 N_i [(_{j-i-1} p_{x+i}) q_{x+j-1}]}{(1+y)(1+x)^{j-i-1}} + \sum_{j=i}^{t-1} \frac{e_{ij}[N_i](y-r_f)}{(1+y)(1+x)^{j-i}} \right. \\
&\quad \left. - \tau \sum_{j=i}^{t-1} \frac{y N_i ({}_{j-i} p_{x+i}) \tilde{V}_j^{(\tau)}}{(1+y)(1+x)^{j-i}} \right\} + \left\{ \sum_{j=i}^{t-1} \frac{f_{ij}[N_i](y-r_f)}{(1+y)(1+x)^{j-i}} - \sum_{j=i}^{t-1} \frac{N_i ({}_{j-i} p_{x+i})}{(1+x)^{j-i}} \right\} \Pi \\
&= c_i[N_i] + d_i[N_i] \Pi
\end{aligned} \tag{37}$$

We solve for the premium that sets $MVL_0[1000] = 0$ or:

$$\Pi = \frac{-c_0[1000]}{d_0[1000]} \tag{38}$$

If tax reserves are equal to equivalence principle reserves at 6%, we find a premium of 1234.95 per person. This is in the neighborhood of the equivalence principle premium of 1203.30. Table VII shows the expected values of the balance sheet items, and Table VIII shows the expected income statement items. The expected value of the evaluation reserves can be calculated from the expected values of other quantities. At time i , the exact value $V_i^{(e)}[N_i]$ is needed and it can be calculated from expected values of other quantities conditional on N_i .

The exact method for determining tax reserves has a noticeable effect on the premium charged. If tax reserves are lowered, this causes taxes to be collected sooner and premiums would need to rise to compensate. If tax reserves are computed at 6.5%, the premium would need to be 1272.80 per person. If tax reserves were reduced still further by using a full preliminary term method at 6.5%, the premium would rise to 1301.37.

In addition, the exact solvency standard for required assets matters but not as much. Reducing the solvency probability to 99% changes the premium to 1233.50, and lowering to 95% changes the premium to 1229.28.

This whole life product can be used to demonstrate the problem if reserves and capital are calculated as something other than $V_i^{(e)}$ and $A_i - V_i^{(e)} - P_i$. Assume reserves are calculated as equivalence principle reserves at 6% instead. Table IX shows the adjusted income during each year. Even though the product earns 10%, just as it was designed to do, the adjusted income statement shows it losing money in early years and making money in later ones. This could lead to incorrect judgments of product profitability and manager performance.

4.5 Guaranteed Minimum Death Benefits

Our final example is that of Guaranteed Minimum Death Benefit Riders in Variable Annuities. Under this contract, if the individual dies he receives the larger of his current account value and the initial premium. It can be viewed as a sequence of European Put Options with a random exercise time equal to the time of death of the individual. We will assume a constant mortality rate q and constant lapse rate λ . Death benefits in this example are paid at the end of the year. A premium Π times the fund level S is paid at the beginning of each year. The fund before premiums are removed follows a Brownian Motion with mean μ and volatility σ . If the strike $X = 1$, we can find the market value of liabilities from option pricing theory as:

$$MVL(S_i) = \sum_{t=1}^{\infty} q(1-q-\lambda)^{t-1} \left[e^{-r_j t} N(-d_2) - S_i (1-\Pi)^t N(-d_1) \right] - \sum_{t=0}^{\infty} (1-q-\lambda)^t \Pi (1-\Pi)^t S_i \quad (39)$$

where

$$d_1 = \frac{\ln[S_i(1-\Pi)^t] + \left(r + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$$

and

$$d_2 = \frac{\ln[S_i(1-\Pi)^t] + \left(r - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$$

We price the product using standard option pricing, choosing Π such that market value at issue, $MVL(1) = 0$. Now, if $\xi_{S_{i+1}}^{0.995} = S_i e^{\mu - \sigma N^{-1}(0.995)}(1 - \Pi)$ is the 99.5th percentile of S_{i+1} , the fund level in a year, given S_i , the fund level today, we find the assets as:

$$A(S_i) = \frac{(1-q-\lambda)MVL(\xi_{S_{i+1}}^{0.995}) + q\text{Max}(1 - \xi_{S_{i+1}}^{0.995}, 0)(1-\tau)}{[1 + r_f(1-\tau)]} + \frac{[\tilde{V}^{(\tau)}(S_i) - (1-q-\lambda)\tilde{V}^{(\tau)}(\xi_{S_{i+1}}^{0.995})] + S_i\Pi\tau}{[1 + r_f(1-\tau)]} \quad (40)$$

In the rest of the analysis, we will assume tax reserves are zero.

We now apply Equation (15) in expectation, with S_i distributed lognormally. We already know the premium Π from option pricing theory, and Equation (10) is used to determine x , the cost of capital. We assume $r_f = 6\%$, $\tau = 34\%$, $q = 2\%$, $\lambda = 10\%$ and find $\Pi = 0.0009422$ and $x = 12.07\%$. We can also apply Equation (13) to determine the values of the evaluation reserve $V^{(e)}(S_i)$. The results of this analysis are shown in Table X.

5. Conclusions

In this paper, we derived formulas for calculating the premiums that should be charged on policies in a discounted cash flow model with tax reserves and required assets that are determined by regulation and can be arbitrary. More importantly, we determine the unique division of required assets into “reserves” and “capital” that allows the product profitability to be correctly evaluated. That is, the profit after capital charges is zero if the product achieves the return assumed in pricing.

This definition of reserves is different than those that are in common use. Even the philosophy is somewhat different. The usual motivation for setting reserves is to set a mostly realistic but somewhat conservative standard that the company will be able to meet its liability obligations. Required Assets are then set to guarantee solvency at a very high level. The analysis in this paper considers setting Required Assets first to guarantee a high level of solvency. The division of those assets into reserves and capital is not then viewed as a solvency issue. After all, the company is just as safe regardless of which of its assets are identified on a piece of paper as “reserves”. Instead, we view the division of assets into capital and reserves as a tool that allows management to more effectively judge the profitability of their lines of business.

Our examples in Section 4 are rather simple, but the procedure is the same in more complicated products with a larger number of risks. The values of N_i , the number of remaining individuals in the whole life example, and S_i , the fund level in the Guaranteed Minimum Death Benefit example, are really special cases of a general filtration F_i that contains all of the information that can be known at time i and might affect the expected values in Equations (10) and (13). The expected values should be taken condition on the information \mathfrak{F}_i available at that point. Almost any stochastic risk

can be contained in this framework without much additional work. A major exception is stochastic interest rates, and it would be useful to extend this work to allow them.

Appendix A. Derivation of Equations (1)-(3)

The insurer sees the following cash flows. At time 0, the insurer cash flow is $CF_0 = P - A_{0,T}$. At time 1, taxes are paid on the premium, and tax reserves first appear.

The total cash flow, including change in required capital, is

$A_{0,T} - A_{1,T} - P\tau + A_{0,T}r_f(1-\tau) + V_{1,T}^{(\tau)}\tau$. At time $2 \leq i \leq T-1$, the cash flow is

$A_{i-1,T} - A_{i,T} + A_{i-1,T}r_f(1-\tau) + (V_{i,T}^{(\tau)} - V_{i-1,T}^{(\tau)})\tau$ and at time T , the cash flow is

$A_{T-1,T} - L_T(1-\tau) + A_{T-1,T}r_f(1-\tau) - V_{T-1,T}^{(\tau)}\tau$. Taking expected values and discounting, we

find:

$$0 = P - A_{0,T} - \frac{P\tau}{1+x} + \sum_{i=1}^{T-1} \frac{A_{i-1,T} - A_{i,T}}{(1+x)^i} + \sum_{i=1}^{T-1} \frac{A_{i-1,T}r_f(1-\tau)}{(1+x)^i} + \frac{V_{1,T}^{(\tau)}\tau}{(1+x)} + \sum_{i=2}^{T-1} \frac{(V_{i,T}^{(\tau)} - V_{i-1,T}^{(\tau)})\tau}{(1+x)^i} + \frac{A_{T-1,T} - E[L_T(1-\tau)] + A_{T-1,T}r_f(1-\tau) + V_{T-1,T}^{(\tau)}\tau}{(1+x)^T} \quad (\text{A1})$$

Which can be rewritten as:

$$0 = P \left(1 - \frac{\tau}{1+x} \right) + \sum_{i=0}^{T-1} A_{i,T} \left\{ \frac{1}{(1+x)^{i+1}} - \frac{1}{(1+x)^i} \right\} + \sum_{i=0}^{T-1} \frac{A_{i-1,T}r_f(1-\tau)}{(1+x)^{i+1}} + \sum_{i=1}^{T-1} V_{i,T}^{(\tau)}\tau \left\{ \frac{1}{(1+x)^i} - \frac{1}{(1+x)^{i+1}} \right\} - \frac{E[L_T](1-\tau)}{(1+x)^T} \quad (\text{A2})$$

Or

$$P(1+x-\tau) = \sum_{i=0}^{T-1} A_{i,T} \left\{ \frac{x}{(1+x)^i} \right\} - \sum_{i=0}^{T-1} \frac{A_{i-1,T}r_f(1-\tau)}{(1+x)^i} - \sum_{i=1}^{T-1} V_{i,T}^{(\tau)}\tau \left\{ \frac{x}{(1+x)^i} \right\} + \frac{E[L_T](1-\tau)}{(1+x)^T} \quad (\text{A3})$$

Now, with $y \equiv \frac{x}{1-\tau}$, we find $1+x-\tau = (1+y)(1-\tau)$ so:

$$P = \frac{E[L_T]}{(1+y)(1+x)^{T-1}} + \sum_{i=0}^{T-1} \frac{A_{i,T}(y-r_f)}{(1+y)(1+x)^i} - \tau \sum_{i=1}^{T-1} V_{i,T}^{(\tau)} \frac{y}{(1+y)(1+x)^i} \quad (\text{A4})$$

Now, defining $A_{i,T} = \frac{E[L_T]}{(1+r_f)^{T-i}} + \Delta A_{i,T}$ and $V_{i,T}^{(\tau)} = \frac{E[L_T]}{(1+r_f)^{T-i}} + \Delta V_{i,T}^{(\tau)}$ we get:

$$P = \frac{E[L_T]}{(1+y)(1+x)^{T-1}} + \frac{E[L_T](y-r_f)}{(1+y)(1+r_f)^T} \sum_{i=0}^{T-1} \left(\frac{1+r_f}{1+x} \right)^i + \sum_{i=0}^{T-1} \frac{\Delta A_{i,T}(y-r_f)}{(1+y)(1+x)^i} - \frac{\tau E[L_T]}{(1+y)(1+r_f)^T} \sum_{i=1}^{T-1} \left(\frac{1+r_f}{1+x} \right)^i - \tau \sum_{i=1}^{T-1} \Delta V_{i,T}^{(\tau)} \frac{y}{(1+y)(1+x)^i} \quad (\text{A5})$$

The geometric series can be summed, and we get:

$$P = \frac{E[L_T]}{(1+y)} \left\{ \frac{1}{(1+x)^{T-1}} + \frac{y-r_f}{(1+r_f)^T} \left[\frac{1 - \left(\frac{1+r_f}{1+x} \right)^T}{1 - \left(\frac{1+r_f}{1+x} \right)} \right] - \frac{\tau y}{(1+r_f)^T} \left[\frac{1 - \left(\frac{1+r_f}{1+x} \right)^T}{1 - \left(\frac{1+r_f}{1+x} \right)} - 1 \right] \right\} + \sum_{i=0}^{T-1} \frac{\Delta A_{i,T}(y-r_f)}{(1+y)(1+x)^i} - \tau \sum_{i=1}^{T-1} \Delta V_{i,T}^{(\tau)} \frac{y}{(1+y)(1+x)^i} \quad (\text{A6})$$

Or

$$P = \frac{E[L_T]}{(1+r_f)^T} \left\{ \frac{(1+r_f)^T}{(1+x)^{T-1}(1+y)} + \frac{y-r_f}{(1+x)^{T-1}(1+y)} \left[\frac{(1+x)^T - (1+r_f)^T}{x-r_f} \right] - \frac{\tau y}{(1+x)^{T-1}(1+y)} \left[\frac{(1+x)^T - (1+r_f)^T}{x-r_f} \right] + \frac{\tau y}{(1+y)} \right\} + \sum_{i=0}^{T-1} \frac{\Delta A_{i,T}(y-r_f)}{(1+y)(1+x)^i} - \tau \sum_{i=1}^{T-1} \Delta V_{i,T}^{(\tau)} \frac{y}{(1+y)(1+x)^i} \quad (\text{A7})$$

And

$$\begin{aligned}
P = & \frac{E[L_T]}{(1+r_f)^T} \left\{ \frac{y-x}{(1+x)^{T-1}(1+y)} \left[\frac{(1+x)^T - (1+r_f)^T}{x-r_f} \right] + \frac{(1+x)^T}{(1+x)^{T-1}(1+y)} \right. \\
& \left. - \frac{\tau y}{(1+x)^{T-1}(1+y)} \left[\frac{(1+x)^T - (1+r_f)^T}{x-r_f} \right] + \frac{\tau y}{(1+y)} \right\} \\
& + \sum_{i=0}^{T-1} \frac{\Delta A_{i,T}(y-r_f)}{(1+y)(1+x)^i} - \tau \sum_{i=1}^{T-1} \Delta V_{i,T}^{(\tau)} \frac{y}{(1+y)(1+x)^i}
\end{aligned} \tag{A8}$$

Simplifying further:

$$\begin{aligned}
P = & \frac{E[L_T]}{(1+r_f)^T} \left\{ \frac{y-x-\tau y}{(1+x)^{T-1}(1+y)} \left[\frac{(1+x)^T - (1+r_f)^T}{x-r_f} \right] + \frac{1+x+\tau y}{1+y} \right\} \\
& + \sum_{i=0}^{T-1} \frac{\Delta A_{i,T}(y-r_f)}{(1+y)(1+x)^i} - \tau \sum_{i=1}^{T-1} \Delta V_{i,T}^{(\tau)} \frac{y}{(1+y)(1+x)^i}
\end{aligned} \tag{A9}$$

Now $x + \tau y = y$ so, finally:

$$P = \frac{E[L_T]}{(1+r_f)^T} + \sum_{i=0}^{T-1} \frac{\Delta A_{i,T}(y-r_f)}{(1+y)(1+x)^i} - \tau \sum_{i=1}^{T-1} \Delta V_{i,T}^{(\tau)} \frac{y}{(1+y)(1+x)^i} \tag{A10}$$

Appendix B. Derivation of Equations (7)-(9)

We prove Equation (7) by backward induction. First, Equation (7) reduces to Equation (5) for $i = T - 1$. Next, we show that if

$$A_{i,T} r_f (1 - \tau) + (V_{i,T}^{(e)} - V_{i+1,T}^{(e)}) - (V_{i,T}^{(\tau)} - V_{i+1,T}^{(\tau)}) \tau - (A_{i,T} - V_{i,T}^{(e)}) x = 0 \quad (\text{B1})$$

and Equation (7) holds for $i + 1$, that is

$$V_{i+1,T}^{(e)} = \frac{E[L_T]}{(1+x)^{T-i-1}} + \sum_{j=i+1}^{T-1} \frac{A_{j,T} [x - r_f (1 - \tau)]}{(1+x)^{j-i}} - \tau \sum_{j=i+1}^{T-1} \frac{(V_{j+1,T}^{(\tau)} - V_{j,T}^{(\tau)})}{(1+x)^{j-i}} \quad (\text{B2})$$

Then

$$V_{i,T}^{(e)} = \frac{E[L_T]}{(1+x)^{T-i}} + \sum_{j=i}^{T-1} \frac{A_{j,T} [x - r_f (1 - \tau)]}{(1+x)^{j-i+1}} - \tau \sum_{j=i}^{T-1} \frac{(V_{j+1,T}^{(\tau)} - V_{j,T}^{(\tau)})}{(1+x)^{j-i+1}} \quad (\text{B3})$$

First, Equation (B1) solves for

$$V_{i,T}^{(e)} = \frac{A_{i,T} [x - r_f (1 - \tau)]}{(1+x)} - \tau \frac{(V_{i+1,T}^{(\tau)} - V_{i,T}^{(\tau)})}{(1+x)} + \frac{V_{i+1,T}^{(e)}}{(1+x)} \quad (\text{B4})$$

Substituting Equation (B2) into Equation (B4) gives:

$$\begin{aligned} V_{i,T}^{(e)} &= \frac{E[L_T]}{(1+x)^{T-i-1} (1+x)} + \frac{A_{i,T} [x - r_f (1 - \tau)]}{(1+x)} + \sum_{j=i+1}^{T-1} \frac{A_{j,T} [x - r_f (1 - \tau)]}{(1+x)^{j-i} (1+x)} \\ &\quad - \tau \frac{(V_{i+1,T}^{(\tau)} - V_{i,T}^{(\tau)})}{(1+x)} - \tau \sum_{j=i+1}^{T-1} \frac{(V_{j+1,T}^{(\tau)} - V_{j,T}^{(\tau)})}{(1+x)^{j-i} (1+x)} \end{aligned} \quad (\text{B5})$$

which combines to give Equation (B3), which is Equation (7) in the main paper.

Next, we prove Equation (8) by noting that for $i = 0$,

$$P(1 - \tau) + A_{0,T} r_f (1 - \tau) - V_{1,T}^{(e)} + V_{1,T}^{(\tau)} \tau - (A_{0,T} - V_{0,T}^{(e)}) x = 0 \quad (\text{B6})$$

if $V_{0,T}^{(e)}$ is the reserve just after the premium is paid. This solves for

$$V_{0,T}^{(e)} = -\frac{P(1-\tau)}{x} + \frac{A_{0,T}[x-r_f(1-\tau)]}{x} - \tau \frac{V_{1,T}^{(\tau)}}{x} + \frac{V_{1,T}^{(e)}}{x} \quad (\text{B7})$$

Substituting Equation (B3) for $V_{1,T}^{(e)}$ gives

$$V_{0,T}^{(e)} = -\frac{P(1-\tau)}{x} - \tau \frac{V_{1,T}^{(\tau)}}{x} + \frac{E[L_T]}{x(1+x)^{T-1}} - \tau \sum_{j=1}^{T-1} \frac{(V_{j+1,T}^{(\tau)} - V_{j,T}^{(\tau)})}{x(1+x)^j} + \sum_{j=0}^{T-1} \frac{A_{j,T}[x-r_f(1-\tau)]}{x(1+x)^j} \quad (\text{B8})$$

Combining the second, third and fourth term gives

$$V_{0,T}^{(e)} = -\frac{P(1-\tau)}{x} + \frac{E[L_T](1-\tau)}{x(1+x)^{T-1}} - \tau \sum_{j=1}^{T-1} \frac{V_{j,T}^{(\tau)}}{(1+x)^j} + \sum_{j=0}^{T-1} \frac{A_{j,T}[x-r_f(1-\tau)]}{x(1+x)^j} \quad (\text{B9})$$

The last three terms can be seen from Equation (1) to be $\frac{P(1+y)(1-\tau)}{x}$ so

$$V_{0,T}^{(e)} = -\frac{P(1-\tau)}{x} + \frac{P(1+y)(1-\tau)}{x} = \frac{Py(1-\tau)}{x} = \frac{Px}{x} = P \quad (\text{B10})$$

Now, we prove Equation (9) from Equation (7) assuming $A_{i,T} = \frac{E[L_T]}{(1+r_f)^{T-i}} + \Delta A_{i,T}$ and

$V_{i,T}^{(\tau)} = \frac{E[L_T]}{(1+r_f)^{T-i}} + \Delta V_{i,T}^{(\tau)}$. Substituting gives:

$$\begin{aligned} V_{i,T}^{(e)} &= \frac{E[L_T]}{(1+x)^{T-i}} + \sum_{j=i}^{T-1} \frac{E[L_T][x-r_f(1-\tau)]}{(1+r_f)^{T-j}(1+x)^{j-i+1}} + \sum_{j=i}^{T-1} \frac{\Delta A_{j,T}[x-r_f(1-\tau)]}{(1+x)^{j-i+1}} \\ &\quad - \tau \sum_{j=i}^{T-1} \frac{E[L_T]}{(1+r_f)^{T-j-1}(1+x)^{j-i+1}} + \tau \sum_{j=i}^{T-1} \frac{E[L_T]}{(1+r_f)^{T-j}(1+x)^{j-i+1}} \\ &\quad - \tau \sum_{j=i}^{T-1} \frac{(\Delta V_{j+1,T}^{(\tau)} - \Delta V_{j,T}^{(\tau)})}{(1+x)^{j-i+1}} \end{aligned} \quad (\text{B11})$$

The sums in the second, fourth and fifth terms are geometric series, and summing them gives:

$$\begin{aligned}
V_{i,t}^{(e)} &= \frac{E[L_T]}{(1+x)^{T-i}} + \frac{E[L_T][x-r_f(1-\tau)]}{(1+r_f)^{T-i}(1+x)^{T-i}} \left[\frac{(1+x)^{T-i} - (1+r_f)^{T-i}}{x-r_f} \right] + \sum_{j=i}^{T-1} \frac{\Delta A_{j,T}[x-r_f(1-\tau)]}{(1+x)^{j-i+1}} \\
&\quad - \frac{\tau(1+r_f)E[L_T]}{(1+r_f)^{T-i}(1+x)^{T-i}} \left[\frac{(1+x)^{T-i} - (1+r_f)^{T-i}}{x-r_f} \right] \\
&\quad + \frac{\tau E[L_T]}{(1+r_f)^{T-i}(1+x)^{T-i}} \left[\frac{(1+x)^{T-i} - (1+r_f)^{T-i}}{x-r_f} \right] - \tau \sum_{j=i}^{T-1} \frac{(\Delta V_{j+1,T}^{(\tau)} - \Delta V_{j,T}^{(\tau)})}{(1+x)^{j-i+1}} \quad (\text{B12})
\end{aligned}$$

The second, fourth and fifth term sum to $E[L_t] \left[\frac{1}{(1+r_f)^{T-i}} - \frac{1}{(1+x)^{T-i}} \right]$ which cancels

off the first term and leaves:

$$V_{i,T}^{(e)} = \frac{E[L_T]}{(1+r_f)^{T-i}} + \sum_{j=i}^{T-1} \frac{\Delta A_{j,T}[x-r_f(1-\tau)]}{(1+x)^{j-i+1}} - \tau \sum_{j=i}^{T-1} \frac{(\Delta V_{j+1,T}^{(\tau)} - \Delta V_{j,T}^{(\tau)})}{(1+x)^{j-i+1}} \quad (\text{B13})$$

which is Equation (9) in the main text.

Appendix C. Derivation of Equation (11)

We start with:

$$\sum_{i=0}^{T-1} \frac{P_i}{(1+x)^i} = \sum_{i=1}^T \frac{E[L_i]}{(1+y)(1+x)^{i-1}} + \sum_{i=0}^{T-1} \frac{A_i(y-r_f)}{(1+y)(1+x)^i} - \tau \sum_{i=1}^{T-1} V_i^{(\tau)} \frac{y}{(1+y)(1+x)^i} \quad (C1)$$

Then, substituting $A_i = \sum_{j=i+1}^T \frac{E[L_j]}{(1+r_f)^{j-i}} - \sum_{j=i+1}^{T-1} \frac{P_j}{(1+r_f)^{j-i}} + \Delta A_i$ (the middle term is not

present for A_{T-1}) and $V_i^{(\tau)} = \sum_{j=i+1}^T \frac{E[L_j]}{(1+r_f)^{j-i}} - \sum_{j=i}^{T-1} \frac{P_j}{(1+r_f)^{j-i}} + \Delta V_i^{(\tau)}$ and moving the

premium terms to the left hand side of the equation, we find:

$$\begin{aligned} & \sum_{i=0}^{T-1} \frac{P_i}{(1+x)^i} + \sum_{i=0}^{T-2} \sum_{j=i+1}^{T-1} \frac{P_j(y-r_f)}{(1+r_f)^{j-i}(1+y)(1+x)^i} - \sum_{i=1}^{T-1} \sum_{j=i}^{T-1} \frac{\tau y P_j}{(1+r_f)^{j-i}(1+y)(1+x)^i} = \\ & \sum_{i=1}^T \frac{E[L_i]}{(1+y)(1+x)^{i-1}} + \sum_{i=0}^{T-1} \sum_{j=i+1}^T \frac{E[L_j](y-r_f)}{(1+r_f)^{j-i}(1+y)(1+x)^i} \\ & - \sum_{i=1}^{T-1} \sum_{j=i+1}^T \frac{\tau y E[L_j]}{(1+r_f)^{j-i}(1+y)(1+x)^i} \\ & + \sum_{i=0}^{T-1} \frac{\Delta A_i(y-r_f)}{(1+y)(1+x)^i} - \tau \sum_{i=1}^{T-1} \Delta V_i^{(\tau)} \frac{y}{(1+y)(1+x)^i} \end{aligned} \quad (C2)$$

The Right Hand Side becomes:

$$\sum_{i=1}^T \frac{E[L_i]}{(1+r_f)^i} + \sum_{i=0}^{T-1} \frac{\Delta A_i(y-r_f)}{(1+y)(1+x)^i} - \tau \sum_{i=1}^{T-1} \Delta V_i^{(\tau)} \frac{y}{(1+y)(1+x)^i} \quad (C3)$$

by reversing the summation order and following the derivation in appendix A for each

term in the summations. We now reverse the summation order on the left hand side to

give:

$$\sum_{i=0}^{T-1} \frac{P_i}{(1+x)^i} + \sum_{i=0}^{T-2} \sum_{j=i+1}^{T-1} \frac{P_j(y-r_f)}{(1+r_f)^{j-i}(1+y)(1+x)^i} - \sum_{i=1}^{T-1} \sum_{j=i}^{T-1} \frac{\tau y P_j}{(1+r_f)^{j-i}(1+y)(1+x)^i} =$$

$$\sum_{i=0}^{T-1} \frac{P_i}{(1+x)^i} + \sum_{j=1}^{T-1} \sum_{i=0}^{j-1} \frac{P_j (y - r_f)}{(1+r_f)^{j-i} (1+y)(1+x)^i} - \sum_{j=1}^{T-1} \sum_{i=1}^j \frac{\tau y P_j}{(1+r_f)^{j-i} (1+y)(1+x)^i} \quad (C4)$$

The interior sum in the third term can be changed to a sum from 0 to $j-1$ and the dummy indices i and j exchanged in the last two terms to give:

$$L.H.S. = P_0 + \sum_{i=1}^{T-1} P_i \left\{ \frac{1}{(1+x)^i} + \left[\frac{(y - r_f)(1+x) - \tau y (1+r_f)}{(1+r_f)^i (1+y)(1+x)} \right] \sum_{j=0}^{i-1} \left(\frac{1+r_f}{1+x} \right)^j \right\} \quad (C5)$$

Doing the sum as a geometric series gives:

$$L.H.S. = P_0 + \sum_{i=1}^{T-1} P_i \left\{ \frac{1}{(1+x)^i} + \left[\frac{(x - r_f)(1+y)}{(1+r_f)^i (1+y)(1+x)^i} \right] \left[\frac{(1+x)^i - (1+r_f)^i}{x - r_f} \right] \right\} \quad (C6)$$

with appropriate cancellations, this leads to:

$$\sum_{i=0}^{T-1} \frac{P_i}{(1+r_f)^i} = \sum_{i=1}^T \frac{E[L_i]}{(1+r_f)^i} + \sum_{i=0}^{T-1} \frac{\Delta A_i (y - r_f)}{(1+y)(1+x)^i} - \tau \sum_{i=1}^{T-1} \frac{\Delta V_i^{(\tau)} y}{(1+y)(1+x)^i} \quad (C7)$$

which is text equation (11).

Appendix D. Arbitrage Argument for the Market Value of Liabilities

In this section we show the market value at time $T - 1$ of a loss at time T , $MV_{T-1}[L_T]$, is the same as $\tilde{P}_{T-1,T}$, the premium that would be charged at time $T - 1$ to cover a loss at time T . The market value at times $i < T - 1$ would be $\tilde{P}_{i,T}$ under a similar argument, as long as tax treatment in the form of tax reserves is the same for all market players. If it is not, there would be opportunities for tax arbitrage between the more favored market participants and the less favored ones.

Assume the company is at risk at time $T - 1$ for loss L_T . The company would charge $\tilde{P}_{T-1,T}$ for assuming this loss. Now suppose the company could go into the market and buy cash flows equal to the loss for $MV_{T-1}[L_T]$. Assuming equal tax treatment, a cost of $MV_{T-1}[L_T]$ today buys $L_T(1 - \tau) + MV_{T-1}[L_T]\tau$ in one period. The company spends $MV_{T-1}[L_T]$ today, and receives a riskless

$$\begin{aligned} & \tilde{P}_{T-1,T}(1 - \tau) + \tilde{P}_{T-1,T}[r_f(1 - \tau)] - L_T(1 - \tau) + L_T(1 - \tau) + MV_{T-1}[L_T]\tau \\ & = \tilde{P}_{T-1,T}(1 - \tau) + \tilde{P}_{T-1,T}[r_f(1 - \tau)] + MV_{T-1}[L_T]\tau \end{aligned} \quad (D1)$$

in one period. The initial investment must earn the after-tax risk-free rate over the period. Therefore:

$$MV_{T-1}[L_T](1 + r_f(1 - \tau)) = \tilde{P}_{T-1,T}(1 - \tau) + \tilde{P}_{T-1,T}[r_f(1 - \tau)] + MV_{T-1}[L_T]\tau \quad (D2)$$

which solves for

$$MV_{T-1}[L_T] = \tilde{P}_{T-1,T} \quad (D3)$$

Appendix E. Derivation of Equations (21)-(23)

Here we determine the market value of liabilities recursively under Solvency II.

From Equation (3) and the definition of the asset requirements under Solvency II, we find:

$$\begin{aligned}
 MV_j[L_T] &= \frac{\tilde{A}_{j,T}(y-r_f)}{(1+y)} + \frac{A_{j+1,T}(y-r_f)}{(1+y)(1+x)} + \sum_{j=i+2}^{T-1} \frac{A_{i,T}(y-r_f)}{(1+y)(1+x)^{i-j}} \\
 &\quad - \tau \frac{yV_{j+1,T}^{(\tau)}}{(1+y)(1+x)} - \tau \sum_{j=i+2}^{T-1} \frac{yV_{j+1,T}^{(\tau)}}{(1+y)(1+x)^{i-j}} + \frac{E[L_T]}{(1+y)(1+x)^{T-j-1}}
 \end{aligned} \tag{E1}$$

where

$$\tilde{A}_{j,T} = \frac{MV_{j+1}[L_T]}{[1+r_f(1-\tau)]} + \tau \frac{(MV_j[L_T] - V_{j+1,T}^{(\tau)})}{[1+r_f(1-\tau)]} \tag{E2}$$

and

$$A_{j+1,T} = \frac{MV_{j+2}[L_T]}{[1+r_f(1-\tau)]} + \tau \frac{(V_{j+1,T}^{(\tau)} - V_{j+2,T}^{(\tau)})}{[1+r_f(1-\tau)]} \tag{E3}$$

Also,

$$MV_{j+1}[L_T] = \frac{\tilde{A}_{j+1,T}(y-r_f)}{(1+y)} + \sum_{j=i+2}^{T-1} \frac{A_{i,T}(y-r_f)}{(1+y)(1+x)^{i-j-1}} \tag{E4}$$

where

$$\tilde{A}_{j+1,T} = \frac{MV_{j+2}[L_T]}{[1+r_f(1-\tau)]} + \tau \frac{(MV_{j+1}[L_T] - V_{j+2,T}^{(\tau)})}{[1+r_f(1-\tau)]} \tag{E5}$$

so

$$A_{j+1,T} = \tilde{A}_{j+1,T} + \tau \frac{(V_{j+1,T}^{(\tau)} - MV_{j+1}[L_T])}{[1+r_f(1-\tau)]} \tag{E6}$$

Combining Equations (E1)-(E6) yields the following equation for the market

value at time j :

$$\begin{aligned}
MV_j[L_T] &= \frac{MV_{j+1}[L_T](y-r_f)}{(1+y)[1+r_f(1-\tau)]} + \tau \frac{MV_j[L_T](y-r_f)}{(1+y)[1+r_f(1-\tau)]} - \tau \frac{V_{j+1,T}^{(\tau)}(y-r_f)}{(1+y)[1+r_f(1-\tau)]} \\
&+ \tau \frac{V_{j+1,t}^{(\tau)}(y-r_f)}{(1+y)[1+r_f(1-\tau)](1+x)} - \tau \frac{MV_{j+1}[L_T](y-r_f)}{(1+y)[1+r_f(1-\tau)](1+x)} \\
&+ \tau \frac{yV_{j+1,t}^{(\tau)}}{(1+y)(1+x)} + \frac{MV_{j+1}[L_T]}{(1+x)} \tag{E7}
\end{aligned}$$

with the definition

$$r_\tau \equiv \frac{\alpha x}{(1-\tau)(1+x)} = \frac{1+y}{1+x} - 1 \tag{E8}$$

Equation (E7) can solve for

$$MV_j[L_T] = \frac{MV_{j+1}[L_T]}{(1+r_f)}(1+r_\tau) - r_\tau \frac{V_{j+1,T}^{(\tau)}}{(1+r_f)} \tag{E9}$$

This establishes Equations (21) and (22)

Using Equation (E9), Equation (23) can be established by induction. If $j = T - 1$,

then

$$P = MV_{T-1}[L_T] = \left[\frac{E[L_T] + (\xi_{L_T}^{0,995} - E[L_T])R}{(1+r_f)} \right] \tag{E10}$$

Now, if

$$\begin{aligned}
MV_{j+1}[L_t] &= \left[\frac{E[L_T] + (\xi_{L_T}^{0,995} - E[L_T])R}{(1+r_f)^{T-j-1}} \right] (1+r_\tau)^{T-j-2} \\
&- r_\tau \sum_{i=j+2}^{T-1} \frac{V_{i,T}^{(\tau)}}{(1+r_f)^{i-j-1}} (1+r_\tau)^{i-j-2} \tag{E11}
\end{aligned}$$

then, substitution into (E9) yields:

$$MV_j[L_t] = \left[\frac{E[L_T] + (\xi_{L_t}^{0.995} - E[L_T])R}{(1+r_f)^{T-j}} \right] (1+r_\tau)^{T-j-1} - r_\tau \sum_{i=j+1}^{T-1} \frac{V_{i,T}^{(\tau)}}{(1+r_f)^{i-j}} (1+r_\tau)^{i-j-1} \quad (\text{E12})$$

Finally, for $j = 0$

$$MV_0[L_T] = \left[\frac{E[L_T] + (\xi_{L_t}^{0.995} - E[L_T])R}{(1+r_f)^t} \right] (1+r_\tau)^{T-1} - r_\tau \sum_{i=1}^{T-1} \frac{V_{i,T}^{(\tau)}}{(1+r_f)^i} (1+r_\tau)^{i-1} \quad (\text{E13})$$

which is Equation (23).

Table I

Balance Sheet Items for Single Premium Single Loss Example

| Time i | $V_i^{(\tau)}$ | $\Delta V_i^{(\tau)}$ | A_i | ΔA_i | $V_i^{(e)}$ | Capital | MVL_i |
|----------|----------------|-----------------------|--------|--------------|-------------|---------|---------|
| 0 | 0.00 | | 392.81 | 19.18 | 385.18 | 7.62 | 0.00 |
| 1 | 381.45 | (14.60) | 405.34 | 9.30 | 398.71 | 6.64 | 407.09 |
| 2 | 408.15 | (11.66) | 428.73 | 8.92 | 423.17 | 5.56 | 430.47 |
| 3 | 436.72 | (8.28) | 453.70 | 8.70 | 449.31 | 4.39 | 455.42 |
| 4 | 467.29 | (4.41) | 597.23 | 125.53 | 477.23 | 120.00 | 482.06 |

Table II

Income Statement Items for Single Premium Single Loss Example

| Time i | Cash Flow CF_i | Cash Income | Change in $V_i^{(e)}$ | Capital Charges |
|----------|------------------|-------------|-----------------------|-----------------|
| 0 | (7.62) | | | |
| 1 | 1.75 | 399.47 | (398.71) | (0.76) |
| 2 | 1.74 | 25.13 | (24.47) | (0.66) |
| 3 | 1.73 | 26.69 | (26.14) | (0.56) |
| 4 | (115.17) | 28.36 | (27.92) | (0.44) |
| 5 | 132.00 | (465.23) | 477.23 | (12.00) |

Table III

Balance Sheet Items for Two Premium Two Loss Example

| Time i | $V_i^{(\tau)}$ | $\Delta V_i^{(\tau)}$ | A_i | ΔA_i | $V_i^{(e)}$ | Capital | MVL_i |
|----------|----------------|-----------------------|--------|--------------|-------------|---------|---------|
| 0 | 0.00 | 0.00 | 491.69 | 75.85 | 0.00 | 60.78 | 0.00 |
| 1 | 48.31 | 7.52 | 601.13 | 129.43 | 50.22 | 120.00 | 51.07 |

Table IV

Income Statement Items for Two Premium Two Loss Example

| Time i | Cash Flow CF_i | Cash Income | Change in $V_i^{(e)}$ | Capital Charges |
|----------|------------------|-------------|-----------------------|-----------------|
| 0 | (60.78) | | | |
| 1 | (53.15) | 56.30 | (50.22) | (6.08) |
| 2 | 132.00 | (38.22) | 50.22 | (12.00) |

Table V

Balance Sheet Items for Term Life Example

| Time i | $E[V_i^{(\tau)}]$ | $E[\Delta V_i^{(\tau)}]$ | $E[A_i]$ | $E[\Delta A_i]$ | $E[V_i^{(e)}]$ | Expected Capital | $E[MVL_i]$ |
|----------|-------------------|--------------------------|--------------|-----------------|----------------|------------------|------------|
| 0 | 0.00 | 0.00 | 2,970,357.36 | 923,348.50 | 0.00 | 785,162.09 | 0.00 |
| 1 | 0.00 | (169,829.39) | 3,112,684.37 | 801,363.61 | 161,338.82 | 809,854.19 | 233,516.71 |

Table VI

Income Statement Items for Term Life Example

| Time i | Cash Flow $E[CF_i]$ | Expected Cash Income | Change in $E[V_i^{(e)}]$ | Expected Capital Charges |
|----------|---------------------|----------------------|--------------------------|--------------------------|
| 0 | (785,162.09) | | | |
| 1 | 53,824.11 | 239,855.03 | (161,338.82) | (78,516.21) |
| 2 | 890,839.61 | (80,353.40) | 161,338.82 | (80,985.42) |

Table VII

Balance Sheet Items for Whole Life Example

| Time i | $E[V_i^{(r)}]$ | $E[\Delta V_i^{(r)}]$ | $E[A_i]$ | $E[\Delta A_i]$ | $E[V_i^{(e)}]$ | Expected Capital | $E[MVL_i]$ |
|----------|----------------|-----------------------|------------|-----------------|----------------|------------------|------------|
| 0 | 0 | 461,087 | 2,118,791 | 1,344,932 | 0 | 883,845 | 0 |
| 1 | 973,497 | 455,206 | 3,097,556 | 1,348,048 | 942,253 | 924,086 | 928,276 |
| 2 | 1,975,545 | 449,073 | 4,042,947 | 1,289,309 | 1,909,323 | 906,458 | 1,879,697 |
| 3 | 3,007,781 | 442,682 | 5,068,854 | 1,280,958 | 2,906,188 | 939,869 | 2,860,738 |
| 4 | 4,068,004 | 436,027 | 6,062,577 | 1,212,534 | 3,927,543 | 916,969 | 3,864,704 |
| 5 | 5,156,873 | 429,101 | 7,133,035 | 1,192,301 | 4,977,302 | 942,771 | 4,896,968 |
| 6 | 6,272,176 | 421,898 | 8,172,941 | 1,115,220 | 6,050,297 | 915,202 | 5,951,035 |
| 7 | 7,414,553 | 414,412 | 9,286,648 | 1,085,005 | 7,150,257 | 934,889 | 7,032,020 |
| 8 | 8,582,788 | 406,639 | 10,375,458 | 1,004,199 | 8,273,100 | 907,249 | 8,134,555 |
| 9 | 9,776,624 | 398,573 | 11,523,999 | 957,699 | 9,421,300 | 914,449 | 9,262,339 |
| 10 | 10,992,970 | 390,210 | 12,690,940 | 907,309 | 10,589,646 | 920,423 | 10,409,212 |
| 11 | 12,230,575 | 381,545 | 13,851,793 | 829,816 | 11,776,857 | 901,989 | 11,573,878 |
| 12 | 13,482,520 | 372,575 | 15,008,548 | 734,218 | 12,977,384 | 866,779 | 12,751,402 |
| 13 | 14,743,571 | 363,300 | 16,187,923 | 652,536 | 14,186,954 | 845,853 | 13,937,940 |
| 14 | 16,006,534 | 353,720 | 17,361,137 | 563,268 | 15,397,495 | 818,587 | 15,125,029 |
| 15 | 17,263,167 | 343,839 | 18,524,330 | 470,894 | 16,601,096 | 789,126 | 16,304,906 |
| 16 | 18,508,796 | 333,662 | 19,675,277 | 377,909 | 17,793,176 | 759,866 | 17,473,031 |
| 17 | 19,737,002 | 323,197 | 20,798,866 | 275,688 | 18,967,267 | 722,225 | 18,622,912 |
| 18 | 20,945,024 | 312,454 | 21,891,614 | 163,526 | 20,121,075 | 675,021 | 19,752,466 |
| 19 | 22,127,650 | 301,442 | 22,950,212 | 43,375 | 21,249,942 | 619,640 | 20,857,283 |
| 20 | 23,278,985 | 290,174 | 23,970,377 | -83,102 | 22,348,468 | 557,241 | 21,932,184 |
| 21 | 24,389,068 | 278,664 | 24,949,569 | -208,383 | 23,407,132 | 494,888 | 22,967,845 |
| 22 | 25,446,519 | 266,928 | 25,861,915 | -346,851 | 24,414,589 | 418,152 | 23,952,936 |
| 23 | 26,437,030 | 254,987 | 26,700,374 | -491,094 | 25,357,430 | 333,519 | 24,874,452 |
| 24 | 27,344,412 | 242,866 | 27,445,820 | -643,892 | 26,220,013 | 237,641 | 25,716,992 |
| 25 | 28,154,099 | 230,595 | 28,088,816 | -799,987 | 26,988,536 | 134,980 | 26,467,100 |
| 26 | 28,853,378 | 218,209 | 28,618,666 | -957,265 | 27,650,812 | 27,092 | 27,112,822 |
| 27 | 29,434,665 | 205,747 | 29,029,711 | -1,113,769 | 28,199,708 | -84,559 | 27,647,227 |
| 28 | 29,891,048 | 193,248 | 29,315,818 | -1,268,705 | 28,628,691 | -199,595 | 28,063,952 |
| 29 | 30,217,224 | 180,756 | 29,470,833 | -1,422,927 | 28,932,830 | -319,290 | 28,358,233 |
| 30 | 30,404,800 | 168,314 | 29,483,813 | -1,578,957 | 29,104,199 | -446,670 | 28,522,351 |
| 31 | 30,438,949 | 155,967 | 29,339,032 | -1,737,587 | 29,128,637 | -583,243 | 28,542,445 |
| 32 | 30,302,311 | 143,767 | 29,018,942 | -1,898,875 | 28,989,567 | -729,898 | 28,402,287 |
| 33 | 29,975,023 | 131,768 | 28,505,687 | -2,060,663 | 28,667,993 | -885,400 | 28,083,269 |
| 34 | 29,438,150 | 120,031 | 27,788,654 | -2,214,495 | 28,145,804 | -1,042,181 | 27,567,649 |

| | | | | | | | |
|----|------------|---------|------------|------------|------------|------------|------------|
| 35 | 28,684,138 | 108,625 | 26,868,922 | -2,351,760 | 27,415,845 | -1,192,091 | 26,848,450 |
| 36 | 27,718,086 | 97,617 | 25,759,230 | -2,464,994 | 26,483,137 | -1,327,662 | 25,930,660 |
| 37 | 26,556,599 | 87,073 | 24,481,838 | -2,548,859 | 25,363,761 | -1,443,095 | 24,830,123 |
| 38 | 25,225,183 | 77,054 | 23,064,883 | -2,601,140 | 24,082,333 | -1,535,344 | 23,571,057 |
| 39 | 23,755,114 | 67,609 | 21,536,077 | -2,625,871 | 22,669,026 | -1,607,392 | 22,183,145 |
| 40 | 22,172,473 | 58,778 | 19,916,998 | -2,627,943 | 21,148,969 | -1,663,216 | 20,691,086 |
| 41 | 20,496,720 | 50,590 | 18,225,467 | -2,609,284 | 19,540,875 | -1,704,029 | 19,113,260 |
| 42 | 18,745,656 | 43,069 | 16,477,597 | -2,571,842 | 17,861,850 | -1,731,105 | 17,466,463 |
| 43 | 16,935,507 | 36,231 | 14,694,136 | -2,511,323 | 16,127,479 | -1,739,527 | 15,765,993 |
| 44 | 15,087,891 | 30,088 | 12,903,664 | -2,421,052 | 14,358,422 | -1,721,671 | 14,032,081 |
| 45 | 13,237,579 | 24,642 | 11,144,263 | -2,298,152 | 12,587,998 | -1,673,213 | 12,297,396 |
| 46 | 11,426,728 | 19,887 | 9,458,476 | -2,142,744 | 10,856,599 | -1,592,503 | 10,601,542 |
| 47 | 9,698,855 | 15,801 | 7,882,648 | -1,962,500 | 9,205,821 | -1,485,267 | 8,985,253 |
| 48 | 8,091,492 | 12,345 | 6,445,844 | -1,766,293 | 7,671,550 | -1,358,697 | 7,483,682 |
| 49 | 6,633,025 | 9,474 | 5,167,641 | -1,563,198 | 6,280,903 | -1,220,550 | 6,123,375 |
| 50 | 5,340,958 | 7,128 | 4,060,675 | -1,358,203 | 5,050,537 | -1,074,909 | 4,920,612 |
| 51 | 4,221,977 | 5,245 | 3,118,860 | -1,164,058 | 3,986,504 | -933,831 | 3,881,161 |
| 52 | 3,273,563 | 3,762 | 2,334,483 | -985,820 | 3,086,356 | -802,374 | 3,002,605 |
| 53 | 2,485,679 | 2,616 | 1,700,907 | -819,858 | 2,340,427 | -677,222 | 2,275,446 |
| 54 | 1,843,024 | 1,748 | 1,192,391 | -676,327 | 1,733,600 | -568,652 | 1,684,648 |
| 55 | 1,324,399 | 1,108 | 792,406 | -550,207 | 1,245,481 | -472,397 | 1,210,175 |
| 56 | 907,546 | 649 | 478,738 | -441,106 | 854,382 | -388,591 | 830,598 |
| 57 | 572,216 | 337 | 243,753 | -336,095 | 540,650 | -304,865 | 526,528 |
| 58 | 304,941 | 140 | 114,551 | -194,391 | 290,681 | -180,272 | 284,302 |
| 59 | 106,836 | 36 | 38,347 | -69,870 | 102,975 | -66,044 | 101,247 |

Table VIII

Income Statement Items for Whole Life Example

| Time i | Cash Flow $E[CF_i]$ | Expected Cash Income | Change in $E[V_i^{(e)}]$ | Expected Capital Charges |
|----------|---------------------|----------------------|--------------------------|--------------------------|
| 0 | (883,845) | | | |
| 1 | 48,144 | 1,030,638 | (942,253) | (88,385) |
| 2 | 110,037 | 1,059,478 | (967,070) | (92,409) |
| 3 | 57,235 | 1,087,511 | (996,865) | (90,646) |
| 4 | 116,886 | 1,115,341 | (1,021,355) | (93,987) |
| 5 | 65,895 | 1,141,457 | (1,049,760) | (91,697) |
| 6 | 121,847 | 1,167,272 | (1,072,994) | (94,277) |
| 7 | 71,833 | 1,191,481 | (1,099,961) | (91,520) |
| 8 | 121,129 | 1,216,331 | (1,122,842) | (93,489) |
| 9 | 83,524 | 1,238,925 | (1,148,200) | (90,725) |
| 10 | 85,471 | 1,259,791 | (1,168,346) | (91,445) |
| 11 | 110,477 | 1,279,254 | (1,187,211) | (92,042) |
| 12 | 125,408 | 1,290,726 | (1,200,527) | (90,199) |
| 13 | 107,604 | 1,296,247 | (1,209,569) | (86,678) |
| 14 | 111,852 | 1,295,126 | (1,210,541) | (84,585) |
| 15 | 111,320 | 1,285,460 | (1,203,601) | (81,859) |
| 16 | 108,172 | 1,270,993 | (1,192,081) | (78,913) |
| 17 | 113,627 | 1,250,077 | (1,174,091) | (75,987) |
| 18 | 119,427 | 1,226,030 | (1,153,808) | (72,223) |
| 19 | 122,883 | 1,196,369 | (1,128,867) | (67,502) |
| 20 | 124,363 | 1,160,490 | (1,098,526) | (61,964) |
| 21 | 118,077 | 1,114,388 | (1,058,664) | (55,724) |
| 22 | 126,225 | 1,056,945 | (1,007,457) | (49,489) |
| 23 | 126,448 | 984,657 | (942,842) | (41,815) |
| 24 | 129,230 | 895,935 | (862,583) | (33,352) |
| 25 | 126,425 | 792,287 | (768,523) | (23,764) |
| 26 | 121,386 | 675,774 | (662,276) | (13,498) |
| 27 | 114,360 | 551,605 | (548,896) | (2,709) |
| 28 | 106,581 | 420,526 | (428,982) | 8,456 |
| 29 | 99,735 | 284,180 | (304,140) | 19,960 |
| 30 | 95,451 | 139,440 | (171,369) | 31,929 |
| 31 | 91,906 | (20,229) | (24,438) | 44,667 |
| 32 | 88,331 | (197,394) | 139,070 | 58,324 |
| 33 | 82,512 | (394,563) | 321,574 | 72,990 |
| 34 | 68,240 | (610,729) | 522,189 | 88,540 |

| | | | | |
|----|-----------|-------------|-----------|---------|
| 35 | 45,692 | (834,178) | 729,960 | 104,218 |
| 36 | 16,362 | (1,051,916) | 932,707 | 119,209 |
| 37 | (17,334) | (1,252,142) | 1,119,376 | 132,766 |
| 38 | (52,060) | (1,425,738) | 1,281,428 | 144,309 |
| 39 | (81,486) | (1,566,841) | 1,413,306 | 153,534 |
| 40 | (104,915) | (1,680,796) | 1,520,057 | 160,739 |
| 41 | (125,509) | (1,774,416) | 1,608,094 | 166,322 |
| 42 | (143,327) | (1,849,428) | 1,679,025 | 170,403 |
| 43 | (164,688) | (1,907,481) | 1,734,370 | 173,110 |
| 44 | (191,809) | (1,943,010) | 1,769,057 | 173,953 |
| 45 | (220,625) | (1,942,592) | 1,770,425 | 172,167 |
| 46 | (248,031) | (1,898,720) | 1,731,398 | 167,321 |
| 47 | (266,486) | (1,810,029) | 1,650,779 | 159,250 |
| 48 | (275,097) | (1,682,797) | 1,534,271 | 148,527 |
| 49 | (274,017) | (1,526,517) | 1,390,647 | 135,870 |
| 50 | (267,695) | (1,352,421) | 1,230,366 | 122,055 |
| 51 | (248,570) | (1,171,524) | 1,064,033 | 107,491 |
| 52 | (224,839) | (993,531) | 900,148 | 93,383 |
| 53 | (205,390) | (826,166) | 745,929 | 80,237 |
| 54 | (176,292) | (674,549) | 606,827 | 67,722 |
| 55 | (153,121) | (544,985) | 488,120 | 56,865 |
| 56 | (131,046) | (438,338) | 391,098 | 47,240 |
| 57 | (122,585) | (352,592) | 313,732 | 38,859 |
| 58 | (155,079) | (280,455) | 249,968 | 30,486 |
| 59 | (132,255) | (205,734) | 187,707 | 18,027 |
| 60 | (72,649) | (109,579) | 102,975 | 6,604 |

Table IX

Adjusted Income Statement Items for Whole Life Example

| Time i | Expected Cash Income | Change in $E[V_i^{(\tau)}]$ | Adjusted Expected Capital Charges | Adjusted Income |
|----------|----------------------|-----------------------------|-----------------------------------|-----------------|
| 0 | | | | |
| 1 | 1,030,638 | (973,497) | (88,385) | (31,244) |
| 2 | 1,059,478 | (1,002,048) | (89,284) | (31,854) |
| 3 | 1,087,511 | (1,032,237) | (84,024) | (28,749) |
| 4 | 1,115,341 | (1,060,223) | (83,828) | (28,709) |
| 5 | 1,141,457 | (1,088,869) | (77,651) | (25,063) |
| 6 | 1,167,272 | (1,115,302) | (76,320) | (24,351) |
| 7 | 1,191,481 | (1,142,378) | (69,332) | (20,229) |
| 8 | 1,216,331 | (1,168,235) | (67,059) | (18,963) |
| 9 | 1,238,925 | (1,193,835) | (59,756) | (14,666) |
| 10 | 1,259,791 | (1,216,346) | (55,913) | (12,468) |
| 11 | 1,279,254 | (1,237,605) | (51,710) | (10,062) |
| 12 | 1,290,726 | (1,251,945) | (44,827) | (6,047) |
| 13 | 1,296,247 | (1,261,051) | (36,164) | (968) |
| 14 | 1,295,126 | (1,262,963) | (28,924) | 3,240 |
| 15 | 1,285,460 | (1,256,633) | (20,955) | 7,872 |
| 16 | 1,270,993 | (1,245,629) | (12,705) | 12,659 |
| 17 | 1,250,077 | (1,228,205) | (4,425) | 17,447 |
| 18 | 1,226,030 | (1,208,023) | 4,751 | 22,759 |
| 19 | 1,196,369 | (1,182,626) | 14,893 | 28,636 |
| 20 | 1,160,490 | (1,151,335) | 25,807 | 34,961 |
| 21 | 1,114,388 | (1,110,082) | 37,328 | 41,634 |
| 22 | 1,056,945 | (1,057,452) | 48,705 | 48,199 |
| 23 | 984,657 | (990,511) | 61,378 | 55,524 |
| 24 | 895,935 | (907,382) | 74,608 | 63,161 |
| 25 | 792,287 | (809,686) | 88,676 | 71,276 |
| 26 | 675,774 | (699,279) | 103,058 | 79,553 |
| 27 | 551,605 | (581,287) | 117,547 | 87,865 |
| 28 | 420,526 | (456,383) | 131,952 | 96,095 |
| 29 | 284,180 | (326,176) | 146,195 | 104,199 |
| 30 | 139,440 | (187,576) | 160,368 | 112,232 |
| 31 | (20,229) | (34,149) | 174,727 | 120,349 |
| 32 | (197,394) | 136,638 | 189,355 | 128,599 |
| 33 | (394,563) | 327,287 | 204,264 | 136,988 |
| 34 | (610,729) | 536,873 | 219,243 | 145,387 |

| | | | | |
|----|-------------|-----------|---------|---------|
| 35 | (834,178) | 754,012 | 233,453 | 153,287 |
| 36 | (1,051,916) | 966,052 | 246,038 | 160,174 |
| 37 | (1,252,142) | 1,161,487 | 256,261 | 165,606 |
| 38 | (1,425,738) | 1,331,416 | 263,593 | 169,271 |
| 39 | (1,566,841) | 1,470,069 | 267,819 | 171,048 |
| 40 | (1,680,796) | 1,582,641 | 269,348 | 171,192 |
| 41 | (1,774,416) | 1,675,753 | 268,672 | 170,009 |
| 42 | (1,849,428) | 1,751,064 | 265,987 | 167,624 |
| 43 | (1,907,481) | 1,810,149 | 261,491 | 164,159 |
| 44 | (1,943,010) | 1,847,616 | 254,755 | 159,362 |
| 45 | (1,942,592) | 1,850,312 | 245,114 | 152,834 |
| 46 | (1,898,720) | 1,810,851 | 232,279 | 144,411 |
| 47 | (1,810,029) | 1,727,874 | 216,263 | 134,108 |
| 48 | (1,682,797) | 1,607,363 | 197,830 | 122,395 |
| 49 | (1,526,517) | 1,458,467 | 177,864 | 109,814 |
| 50 | (1,352,421) | 1,292,067 | 157,267 | 96,913 |
| 51 | (1,171,524) | 1,118,981 | 136,533 | 83,991 |
| 52 | (993,531) | 948,414 | 116,930 | 71,813 |
| 53 | (826,166) | 787,884 | 98,958 | 60,676 |
| 54 | (674,549) | 642,655 | 82,247 | 50,353 |
| 55 | (544,985) | 518,625 | 67,808 | 41,448 |
| 56 | (438,338) | 416,852 | 55,131 | 33,646 |
| 57 | (352,592) | 335,330 | 44,175 | 26,914 |
| 58 | (280,455) | 267,275 | 33,643 | 20,463 |
| 59 | (205,734) | 198,105 | 19,453 | 11,824 |
| 60 | (109,579) | 106,836 | 6,991 | 4,248 |

Table X

Balance Sheet Items for Guaranteed Minimum Death Benefit Example

| Stock S_i | $A(S_i)$ | $V^{(e)}(S_i)$ | Capital | MVL |
|-------------|-----------|----------------|----------|-----------|
| 0.1 | 0.095538 | 0.074538 | 0.021000 | 0.093038 |
| 0.2 | 0.085601 | 0.060506 | 0.025095 | 0.077085 |
| 0.3 | 0.076054 | 0.048143 | 0.027910 | 0.062327 |
| 0.4 | 0.066937 | 0.037314 | 0.029623 | 0.048891 |
| 0.5 | 0.058282 | 0.027922 | 0.030360 | 0.036871 |
| 0.6 | 0.050114 | 0.019893 | 0.030220 | 0.026345 |
| 0.7 | 0.042455 | 0.013171 | 0.029283 | 0.017386 |
| 0.8 | 0.035324 | 0.007702 | 0.027622 | 0.010050 |
| 0.9 | 0.028739 | 0.003377 | 0.025362 | 0.004315 |
| 1.0 | 0.022715 | 0.000000 | 0.022716 | 0.000000 |
| 1.1 | 0.017271 | -0.002663 | 0.019933 | -0.003192 |
| 1.2 | 0.012416 | -0.004818 | 0.017234 | -0.005582 |
| 1.3 | 0.008147 | -0.006621 | 0.014768 | -0.007433 |
| 1.4 | 0.004433 | -0.008174 | 0.012607 | -0.008936 |
| 1.5 | 0.001221 | -0.009547 | 0.010768 | -0.010214 |
| 1.6 | -0.000991 | -0.010746 | 0.009755 | -0.011346 |
| 1.7 | -0.002593 | -0.011828 | 0.009235 | -0.012380 |
| 1.8 | -0.003916 | -0.012836 | 0.008920 | -0.013346 |
| 1.9 | -0.005026 | -0.013789 | 0.008763 | -0.014266 |
| 2.0 | -0.005974 | -0.014700 | 0.008726 | -0.015152 |

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